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Asset Liability Modelling and Pension Schemes: The Application of Robust Optimization to USS

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Abstract
This paper uses a novel numerical optimization technique - robust optimization - that is well suited to solving the asset-liability management (ALM) problem for pension schemes. It requires the estimation of fewer stochastic parameters, reduces estimation risk and adopts a prudent approach to asset allocation. This study is the first to apply it to a real-world pension scheme and to use the Sharpe ratio as the objective of an ALM problem. We also disaggregate the pension liabilities into three components - active members, deferred members and pensioners, and transform the optimal asset allocation into the scheme's projected contribution rate. We extend the robust optimization model to include liabilities, and use it to derive optimal investment policies for the Universities Superannuation Scheme (USS), benchmarked against the same ALM model with deterministic parameters, the Sharpe and Tint model and the actual USS investment decisions. Over a 144 month out-of-sample period we find that robust optimization is superior to the three benchmarks on all the performance criteria.

Keywords
robust optimization, pension scheme, asset-liability model, Sharpe ratio, Sharpe-Tint model

JEL Classifications
G11, G12, G22, G23

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1 Introduction

Pension schemes are among the largest institutional investors, and in 2012 the OECD countries had pension assets of $32.1 trillion (with liabilities several times larger), accounting for 41% of the assets held by institutional investors (OECD, 2013). Pension schemes have long time horizons, with new members likely to be drawing a pension many years later, and therefore need to make long term investment decisions to meet their liabilities. To help them do this many of the larger defined benefit (DB) pension schemes model their assets and liabilities using asset-liability management (ALM). These models determine the scheme's asset allocation, with stock selection left to the fund managers. While a widespread switch to defined contribution schemes is underway, DB schemes will remain very large investors for decades to come, particularly those in the public sector, as they continue to serve their existing members and pensioners. After the equity bubble of the late 1990's, over the past decade DB schemes have faced 'a perfect storm' of low interest rates, low asset returns and increasing longevity. This outcome was much worse than almost any forecast, and resulted in very large deficits. For example, in May 2012 the largest 7,800 UK DB schemes had an aggregate deficit of £312.1 billion, or roughly one fifth of UK annual GDP.

There are a variety of techniques for deriving optimal ALM strategies for pension funds, and they fall into four main categories: stochastic programming, e.g. Kusy and Ziemba (1986), Kouwenberg (2001), Kouwenberg and Zenios (2006) and Geyer and Ziemba (2008); dynamic programming, e.g. Rudolf and Ziemba (2004) and Gao (2008); portfolio theory, e.g. Sharpe and Tint (1990), and stochastic simulation, e.g. Boender (1997). The process of computing optimal ALM strategies can be challenging, and most of the existing techniques are too demanding to be widely applied in practice.

While stochastic programming is the most popular technique for solving ALM problems, the present study uses the new technique of robust optimization. Robust optimization has attracted considerable attention in recent years and is considered by many practitioners and academics to be a powerful and efficient technique for solving problems subject to uncertainty (Ben-Tal and Nemirovski, 1998). It has been applied to portfolio optimization and asset management, allowing for the uncertainty that occurs due to estimation errors in the input parameters, e.g. Ben-Tal and Nemirovski (1999), Rustem et al. (2000), Ceria and Stubbs (2006) and Bertsimas and Pechamanova (2008). Robust optimization recognises that the market parameters of an ALM model are stochastic, but lie within uncertainty sets (e.g. upper and lower limits). Robust optimization requires the specification of the maximum deviation from each of the expected values of the stochastic input parameters the decision maker is prepared to accept. This maximum deviation is set to reflect the level of confidence (denoted by ω, where $0 \leq \omega \leq 1$) the decision maker requires that the optimal value of the objective function will be achieved when the solution is implemented.

Robust optimization has a number of advantages over other analytical techniques. First, it solves the worst-case problem by finding the best outcome in the most unfavourable circumstances (the maximin), i.e. each stochastic parameter is assumed to take the most unfavourable value in its uncertainty set. Since DB schemes must meet their pensions promise, investment strategies based on ALMs using robust optimization are well suited to the pension context where prudence and safety are important. Second, previous techniques such as stochastic programming and stochastic dynamic control are computationally demanding, while robust optimization is much easier to solve. The computational complexity of robust optimization is the same as that of quadratic programming in terms of the number of assets and time periods; while the complexity of scenario based approaches, such as stochastic programming and dynamic stochastic control, is exponential in the number of assets and time periods. Therefore realistic robust optimization problems can be solved in a few seconds of computer processing time. Finally, in comparison with other techniques, robust optimization is less sensitive to errors in the input parameters, i.e. estimation errors. This tends to eliminate extreme (e.g. corner) solutions, leading to investment in more stable and diversified portfolios, with the benefit of lower portfolio transactions costs.

Portfolio theory is highly susceptible to estimation errors that overstate returns and understate risk, and three main approaches to dealing with such errors in portfolio and ALM models have been used. The first approach involves changing the way the mean and covariance matrix of returns are estimated; for example, James-Stein shrinkage estimation (e.g. Jobson, Korkie and Ratti, 1979), Bayes estimation (e.g. Black and Litterman, 1992), and the overall mean of the estimated covariances (Elton and Gruber, 1973). A second approach is to constrain the asset proportions to rule out the extreme solutions generated by the presence of estimation errors (e.g. Board and Sutcliffe, 1995). A third approach is to use simulation to generate many data sets, each of which is used to compute an optimal portfolio, with the average of these optimal portfolios giving the overall solution (Michaud, 1999). Robust optimization offers a fourth approach to estimation errors in which the objective function is altered to try to avoid selecting portfolios that promise good results, possibly due to estimation errors. It adopts a maximin objective function, where the realized outcome has a chosen probability of being at least as good as the optimal robust optimization solution, which should rule out solutions based on favourable estimation errors.

The only previous application of robust optimization to pension schemes is Gulpinar and Pechamanova (2013). Their example uses two asset classes – an equity market index and the risk-free rate, 80 observations, non-stochastic liabilities and an investment horizon that consists of four investment periods of three months each. They imposed a lower bound on the funding ratio (i.e. assets/liabilities) which was constrained to never be less than 90%. The objective was to maximise the expected difference between the terminal value of scheme assets and contributions to the scheme. There was an experimental simulation, rather than out-of-sample testing. The chosen confidence level, i.e. protection against estimation errors, was set to be the same for each uncertainty set, and varied between zero and three. They found a clear negative relationship between the chosen confidence level and the value of the objective function.

We use the robust optimization framework proposed by Goldfarb and Iyengar (2003), which has the advantage that it does not require a covariance matrix to be estimated, but instead involves estimating a factor loadings matrix using multiple regression. We extend the Goldfarb and Iyengar model from portfolio problems to ALM by incorporating risky liabilities with their own fixed 'negative' weights, disaggregate the liabilities into three categories, and include upper and lower bounds on the proportions of assets invested in the main asset classes. The objective function we use is the Sharpe ratio, which gives the solution that maximises the excess return per unit of risk[1], and is the first study to use the Sharpe ratio as the objective for an ALM study of a pension scheme. We compare these results with those of three benchmarks. The first is the same ALM model, but with deterministic parameters (i.e. $\omega = 0$). The second is the actual portfolios chosen by USS. The final benchmark is a modified version of the Sharpe and Tint (1990) model which has been widely used in ALM models (Ang, Chen & Sundaresan, 2013; Chun, Ciochetti & Shilling, 2000; Craft, 2001, 2005; De Groot & Swinkels, 2008; Ezra, 1991; Keel & Muller, 1995; Nijman & Swinkels, 2008). Stochastic programming is not used as a benchmark because of the enormous computational burden this would entail. In our application to USS with 14 assets and liabilities, and assuming five independent outcomes each three year period, the total number of scenarios to be evaluated for the four out-of-sample periods would be $4(5^{14})$, or 24.4 trillion[2].

[1] For mathematical reasons the expected Sharpe ratio must be constrained to be strictly positive, and so the lower bound on expected returns of the asset-liability portfolio is set to 0.1%, rather than zero. This rules out asset allocations that are expected to worsen the scheme's funding position.

[2] We use a single period portfolio model, and there are two alternative theoretical justifications for the use of such models when investors can re-balance their portfolios (Campbell and Viceira, 2002, pp. 33-35).

Our resulting ALM model is applied to the Universities Superannuation Scheme (USS) using monthly data for 18 years (1993 to 2011), i.e. 216 months. The choice of USS has the advantages that, as the sponsors are tax exempt, there is no case for 100% investment in bonds to reap a tax arbitrage profit. In addition, because the sponsors' default risk is uncorrelated with that of USS, there is no need to include the sponsors' assets in the ALM model. The data is adjusted to allow for USS's foreign exchange hedging from April 2006 onwards as this is when USS changed its benchmark onto a sterling basis. Finally, we use an actuarial model to transform the robust optimization solutions (i.e. asset proportions) into the scheme's contribution rate.

Ultimately pension scheme trustees are concerned about the scheme's projected funding ratio and contribution rate. Since the asset allocation has an important influence on the contribution rate and funding ratio, trustees need to make the asset allocation decision in the light of its effect on these two variables. Trustees wish to reduce the cost of the scheme to the sponsor and members by minimising the contribution rate. Trustees are also concerned with the regulatory limits on the funding ratio. UK legislation places upper and lower limits on the funding ratio of pension schemes, and the likelihood of breaching these requirements must be considered when making the asset allocation decision. MacBeth, Emanuel and Heatter (1994) report that pension scheme trustees prefer to make judgements in terms of the scheme's projected funding ratio and contribution rate, rather than the scheme's asset-liability portfolio. We use the model developed by Board and Sutcliffe (2007), which is a generalization of Haberman (1992), to transform the asset allocations to contribution rates. This generalization allows the discount rate to differ from the investment return, which improves the economic realism of the actuarial model.

Assuming all dividends are reinvested, the first justification is that the investor has a logarithmic utility function. The second justification is that asset returns are independently and identically distributed over time, and investor risk aversion is unaffected by changes in wealth. The immediate reinvestment of dividends is common practice, while there is evidence that the share of household liquid assets allocated to risky assets is unaffected by wealth changes, implying constant risk aversion (Brunnermeier and Nagel, 2008). For the 18 years of monthly data used in the empirical application below only three of the 11 asset returns fail to have first, second or third order autocorrelation at the 5% level of significance for the full data period (UK equities, 10 year US bonds and 20 year US bonds). Excluding the last three years, six assets do not have significant first order autocorrelation as EU equities and US equities now fail to have any significant autocorrelation, while commodities have no significant first order autocorrelation. At the 1% level of significance the only assets that exhibit first order autocorrelation for the 1993-2008 period are UK 10 year bonds and UK property. Property index returns are well known to exhibit autocorrelation due to the use of stale prices in their construction. As mentioned, a multi-period model could require evaluating 24.4 trillion solutions, and so be impractical. Therefore a single period model appears to be a reasonable simplification that has been used by many researchers for solving pension ALM problems; including Ang, Chen & Sundaresan (2013); Chun, Ciochetti & Shilling (2000); Craft (2001, 2005); De Groot & Swinkels (2008); Ezra (1991); Keel & Muller (1995); Nijman & Swinkels (2008); Sharpe & Tint (1990).

The remainder of this paper is organized as follows. In section 2 we provide a brief overview of robust optimization and describe our robust optimization ALM model. Section 3 describes USS, and in section 4 we describe the data set, explain how returns on the three liabilities were estimated, and use a factor model to estimate the three uncertainty sets. In section 5 we compute optimal out-of-sample investment policies for USS using robust optimization and the three benchmarks, and the implications of the asset allocations for the contribution rate. Finally, section 6 summarizes our findings.

2 Robust Optimization ALM Model

Robust optimization is a powerful and efficient technique for solving optimization problems subject to parameter uncertainty. In the model used below there are three uncertainty sets corresponding to the three stochastic parameters in the factor model (see equation (1) below).These are for (a) mean asset returns, (b) the coefficients of the factor model used to estimate the factor loadings matrix (which takes the place of the covariance matrix used in conventional portfolio theory), and (c) the disturbances of the factor model. The decision-maker specifies the level of confidence (denoted by ω), that the Sharpe ratio given by the optimal robust optimization solution will be achieved or surpassed in the out-of-sample period. Each uncertainty set is defined by its shape, the chosen confidence level (ω), and the parameters estimated by the factor model. The shape of the uncertainty sets (e.g. ellipsoidal, box, etc.) is chosen to reflect the decision-maker's understanding of the probability distributions of the stochastic parameters. An appropriate selection of the shape of the uncertainty sets is essential if the robust optimization problem is to be tractable (Natarajan, 2009). Elliptical (ellipsoidal) uncertainty sets are very widely used when the constraints involve standard deviations, and in most cases this choice results in a tractable and easy solved problem (Bertsimas, Pachamanova and Sim, 2004).

The initial formulation of the robust optimization problem with its uncertain parameters is transformed into the robust counterpart (or robust analog problem). This transformed problem has certain parameters (the worst-case value within its uncertainty set for each stochastic parameter) with only linear, quadratic and second-order cone constraints. This second order cone problem (SOCP) can be easily solved, see Ben-Tal and Nemirovski (1998, 1999, 2000).

We divide the liabilities into three components: (i) members who are currently contributing to the pension fund (active members), (ii) deferred pensioners who have left the scheme but not yet retired and so currently do not generate any cash flows for the fund, and (iii) pensioners who are currently receiving a pension and so generate cash outflows from the fund. The liability for active members varies principally with salary growth until they retire, and then with interest rates, longevity and inflation. For deferred pensioners, their liability varies with the chosen revaluation rate until they retire (USS uses inflation as the revaluation rate), and then with interest rates, longevity and inflation. Pensions in payment vary with interest rates, longevity and inflation as from the current date. In our extension of the Goldfarb and Iyengar model, the vector of returns becomes a joint vector containing both the `positive' (asset classes) and 'negative' (pension liabilities) returns. Similarly the matrix of factor loadings is extended to include liabilities as well as assets. See Appendix B for details of the three uncertainty sets.

In the long run pension schemes do not take short positions, and so we rule out short selling by imposing non-negativity constraints on the asset proportions. We also do not permit borrowing money because UK pension schemes are prohibited from long-term borrowing. They are also prohibited from using derivatives, except for hedging or facilitating portfolio management. In addition, pension funds usually set upper and lower limits on the proportion of assets invested in asset classes such as equities, fixed income, alternative assets, property and cash. Therefore our ALM model includes upper and lower bounds on broad asset classes to rule out solutions that would be unacceptable to the trustees.

ALM studies of pension schemes using programming models have employed a wide variety of objective functions. For example, maximise the terminal wealth or surplus with penalties for risk and breaching constraints; or minimise the present value of contributions to the scheme with penalties for risk and breaching constraints. While some of the penalties can be quantified in monetary terms, a penalty for risk requires the specification of a risk aversion co-efficient. Many previous ALMs have assumed the pension scheme has a particular utility function of wealth or pension surplus, e.g. constant relative risk aversion (CRRA), constant absolute risk aversion (CARA) or quadratic utility, together with a specified risk aversion parameter. However, pension schemes are non-corporeal entities with an infinite life, and specifying their preferences in terms of a calibrated utility function is problematic.

An alternative approach is to use the market price of risk, which can be calculated from market data. The Capital Asset Pricing Model implies that the market's trade-off between risk (standard deviation) and return is given by the slope of the Capital Market Line. This leads to the use of the Sharpe ratio to select a risky portfolio, and has been widely used in academic studies[3]. Following

[3] Roy's (1952) safety first criterion seeks to minimise the probability of the realised portfolio return falling below some minimum threshold. If the minimum acceptable return is the risk free rate, the Sharpe ratio is equivalent to Roy's safety first criterion.

Sharpe (1994), we define the Sharpe ratio as the return on a fund in excess of that on a benchmark portfolio, divided by the standard deviation of the excess returns. The risk free rate is usually chosen as the benchmark portfolio, but we use the liability portfolio. So in our case the fund is the pension scheme's asset portfolio, and the benchmark is its liability portfolio. Therefore we divide the mean return on the asset-liability portfolio by the standard deviation of the asset-liability portfolio. Individual risk-return preferences will differ from this ratio but, since large pension schemes have well diversified portfolios, it gives the average value of individual preferences revealed in the capital market, and so offers a reasonable objective for a very large multi-employer scheme like USS.

By using robust optimization to solve the problem, depending on the size of the chosen confidence level (ω), the scheme's chosen portfolio is effectively more risk averse than simply maximizing the Sharpe ratio. Use of the Sharpe ratio ignores the scheme's current funding ratio, but if the risk attitudes of pension schemes are wealth independent, the funding ratio is irrelevant. The empirical evidence on the effect of the funding ratio on the asset allocation, allowing for real world influences such as default insurance, taxation and financial slack, is mixed (An, Huang and Zhang, forthcoming; Anantharaman and Lee, forthcoming; Atanasova and Gatev, forthcoming; Bodie, Light, Morck and Taggart, 1985, 1987; Comprix and Muller, 2006; Mohan and Zhang, forthcoming; Li, 2010; McCarthy and Miles, 2013; Munro and Barrie, 2003; Petersen, 1996; Rauh, 2009). This is consistent with the view that pension schemes do not alter their asset allocation in a predictable way with their funding ratio, except when very close to default.

The use of a factor model means there is no need to estimate the covariance matrix of the asset-liability returns, just the covariance matrix of the factor returns. This reduces the dimensionality of the problem from *[n(n−1)/2+n]*, where *n* is the total number of assets and liabilities, to *[m(m−1)/2+m]*, where *m* is the number of factors. In our case with 14 assets and liabilities and four factors, this reduces the number of covariances that must be estimated from 105 to 10.

We follow Goldfarb and Iyengar (2003) by assuming that asset and liability returns are described by the following factor model:-

$$\tilde{\mathbf{r}}_{\mathrm{A,L}} = \tilde{\boldsymbol{\mu}}_{\mathrm{A,L}} + \tilde{\mathbf{V}}^{T}\mathbf{f} + \tilde{\boldsymbol{\varepsilon}}_{\mathrm{A,L}} \tag{1}$$

where $\tilde{\mathbf{r}}_{\mathrm{A,L}}$ is a joint column vector with $n_{\mathrm{A}} + n_{\mathrm{L}}$ elements that contains the uncertain asset and liability returns; $\tilde{\boldsymbol{\mu}}_{\mathrm{A,L}}$, with $n_{\mathrm{A}} + n_{\mathrm{L}}$ elements, is the joint column vector of the random asset and liability mean returns; the column vector $\mathbf{f}$ with m (number of factors) elements contains the factor returns that drive the risky assets and liabilities; the matrix $\tilde{\mathbf{V}}$ with m rows and $n_{\mathrm{A}}+n_{\mathrm{L}}$ columns contains the corresponding uncertain factor coefficients; and $\tilde{\boldsymbol{\varepsilon}}_{\mathrm{A,L}}$ with $n_{\mathrm{A}}+n_{\mathrm{L}}$ elements is the column vector of uncertain disturbances. The square matrix $\mathbf{F}$ with m rows and columns; and the diagonal matrix $\tilde{\mathbf{D}}$ with $n_{\mathrm{A}}+n_{\mathrm{L}}$ elements on its diagonal is the covariance matrix of the column vectors of the uncertain factor returns and disturbances respectively.

The matrix of uncertain factor coefficients $\left(\tilde{\mathbf{V}}\right)$ belongs to an elliptical uncertainty set denoted by S_{υ}, while the components of the column vector of random mean returns $\left(\tilde{\boldsymbol{\mu}}_{\mathrm{A,L}}\right)$ and the column vector that contains the diagonal elements of the covariance matrix of the disturbances $\left(\tilde{\mathbf{D}}\right)$ lie within certain intervals which are represented by the uncertainty structures S_m and S_d. The analytical form of these uncertainty structures, which are completely parameterized by market data and the parameter ω, can be seen in Appendix B. The parameter ω specifies the level of confidence, and hence allows the provision of probabilistic warranties on the performance of the robust portfolios.

The robust optimization problem we solve is given by the following maximin problem:-

$$
\begin{aligned}
&\underset{\boldsymbol{\Phi}_{\mathrm{A}}}{\text{maximize}} \quad \left[\frac{\underset{\left\{\tilde{\boldsymbol{\mu}}_{\mathrm{A,L}}\in S_m\right\}}{min}\left[\tilde{\boldsymbol{\mu}}_{\mathrm{A,L}}^{\mathrm{T}}\boldsymbol{\Phi}_{\mathrm{A,L}}\right]}{\sqrt{\underset{\left\{\left(\tilde{\mathbf{V}}\in S_{\upsilon}\right)\right\}}{max}\left[\boldsymbol{\Phi}_{\mathrm{A,L}}^{\mathrm{T}}\tilde{\mathbf{V}}^{\mathrm{T}}\mathbf{F}\tilde{\mathbf{V}}\boldsymbol{\Phi}_{\mathrm{A,L}}\right]+\underset{\left\{\tilde{\mathbf{D}}\in S_d\right\}}{max}\left[\boldsymbol{\Phi}_{\mathrm{A,L}}^{\mathrm{T}}\tilde{\mathbf{D}}\boldsymbol{\Phi}_{\mathrm{A,L}}\right]}}\right]\\
&s.t: \quad \mathbf{1}^{\mathrm{T}}\boldsymbol{\Phi}_{\mathrm{A}}=1\\
&\qquad\;\; \mathbf{1}^{\mathrm{T}}\boldsymbol{\Phi}_{\mathrm{L}}=-1\\
&\qquad\;\; \Phi_{\mathrm{A},i}\geq 0,\ \forall i\\
&\qquad\;\; -\sum_{i\in U}\Phi_{\mathrm{A},i}+\theta_U\mathbf{1}^{\mathrm{T}}\boldsymbol{\Phi}_{\mathrm{A}}\geq 0\\
&\qquad\;\; \sum_{i\in L}\Phi_{\mathrm{A},i}-\theta_L\mathbf{1}^{\mathrm{T}}\boldsymbol{\Phi}_{\mathrm{A}}\geq 0
\end{aligned}
\tag{2}
$$

where $\boldsymbol{\Phi}_{\mathrm{A,L}}$ denotes the joint column vector of non-negative asset proportions $\left(\boldsymbol{\Phi}_{\mathrm{A}}\right)$ and negative liability proportions $\left(\boldsymbol{\Phi}_{\mathrm{L}}\right)$, while θ_L and θ_U represent the minimum and maximum percentage of the corresponding asset sub-sets U and L. The objective in equation (2) is to maximise the Sharpe ratio under the worst circumstances (i.e. maximin). The worst case mean return in the nominator is divided by the worst case variance (two terms) in the denominator. These worst case expressions (described in Appendices B and C) are functions of the parameter ω (the confidence level). As ω is increased the size of the uncertainty sets increases, which worsens the worst case circumstances. The maximin optimization problem described in equation (2) can be converted to a second order cone program (SOCP), and hence is a tractable and easily solved mathematical optimization problem. Further details of this transformation appear in Appendix C.

The first benchmark we use is the robust optimization model with $\omega = 0$, i.e. no robust optimization, and the second is the actual portfolios chosen by USS. The third benchmark is a modified version of the Sharpe and Tint (1990) model, where the objective function has been changed so that it is now to maximise the expected return on the asset portfolio in excess of the liability portfolio, divided by the standard deviation of these excess returns. This model has the same additional constraints on the asset weights as the robust optimisation and deterministic models, while the covariance matrix is estimated directly from the asset and liability returns. Hence, this benchmark is described as follows:-

$$
\begin{aligned}
&\underset{\Phi_{\mathrm{A}}}{maximize} && \frac{\mathrm{E}\left[\boldsymbol{\Phi}_{\mathrm{A,L}}^{\mathrm{T}}\tilde{\mathbf{r}}_{\mathrm{A,L}}\right]}{\sqrt{\mathrm{Var}\left[\boldsymbol{\Phi}_{\mathrm{A,L}}^{\mathrm{T}}\tilde{\mathbf{r}}_{\mathrm{A,L}}\right]}} \\
&s.t: && \mathbf{1}^{\mathrm{T}}\boldsymbol{\Phi}_{\mathrm{A}} = 1 \\
& && \mathbf{1}^{\mathrm{T}}\boldsymbol{\Phi}_{\mathrm{L}} = -1 \\
& && \Phi_{\mathrm{A},i} \geq 0, \ \forall i \\
& && -\sum_{i \in U}\Phi_{\mathrm{A},i} + \theta_U \mathbf{1}^T\boldsymbol{\Phi}_{\mathrm{A}} \geq 0 \\
& && \sum_{i \in L}\Phi_{\mathrm{A},i} - \theta_L \mathbf{1}^T\boldsymbol{\Phi}_{\mathrm{A}} \geq 0
\end{aligned}
\tag{3}
$$

3 Application to USS

The robust optimization model will be used to derive optimal investment policies for the Universities Superannuation Scheme (USS). USS was created in 1974 as the main pension scheme for academic and senior administrative staff in UK universities and other higher education and research institutions (Logan, 1985). In 2013 USS was the second largest pension scheme in the UK, and the 36th largest in the world with 303,060 active members, deferred pensioners and pensioners. It is a multi-employer scheme with 379 separate sponsors (or institutions), and assets valued at £39 billion in 2013. There are two important advantages in using USS as the real world application. First, there is no need to include the assets and liabilities of the sponsors in the model; and second, because the USS sponsors are tax exempt, the optimal asset allocation is not the corner solution of 100% bonds which maximises the tax relief.

It is generally accepted that a pension scheme and its sponsor should be treated as a single economic entity, and this has a number of important implications for ALMs. The discount rate used in valuing pension liabilities must be increased above the riskless rate to reflect the risk of default by the sponsor. This sponsor default risk can be reduced by incorporating the assets and liabilities of the sponsor in the ALM model, allowing for correlations between the assets and liabilities of the sponsor and the pension scheme. For example, if the sponsor makes cars, investment by the pension scheme in the shares of car producers increases the risk of default because, when the scheme has a deficit due to poor investment returns, the sponsor is also likely to be experiencing adverse business conditions. The value of the pension liabilities is an input to the ALM model, and this valuation depends on the discount rate used to value the liabilities, which in turn depends on the risk of default by the sponsor. The risk of default depends on the asset allocation of the pension scheme, which is the output from the ALM model. This leads to a circularity that Inkmann and Blake (2012) solve using simulation.

However, USS is a multi-employer scheme where the 379 institutions (sponsors) are funded largely by the UK government and student fees. Therefore the default risk of the sponsor is effectively uncorrelated with the assets and liabilities of the scheme. In addition, USS is a last-man-standing multi-employer scheme, and default would require the bankruptcy of every institution, i.e. the collapse of the UK university and research community. Therefore, for USS the default risk of the sponsor is both very low and independent of that of the scheme, and so need not be considered when setting the discount rate. Therefore the assets and liabilities of the 379 sponsors will not be incorporated in the ALM model of USS.

UK pension schemes are tax exempt and so there is no need to adjust their returns for taxation. However, taxation of the sponsor creates an arbitrage argument for a pension fund to invest 100% in bonds (as did Boots in 2000). There are two different arguments for 100% bond investment by a pension fund:- (a) Black (1980) (see also Surz, 1981; Black and Dewhurst, 1981, Frank, 2002), and Tepper (1981) (see also Bader, 2003, Frank, 2002). However, UK universities (the sponsors of USS) are not liable to pay tax, and so the tax arguments of Black and Tepper leading to 100% investment in bonds do not apply.

4 Data and Analysis

4.1 Assets

The main asset classes used by the trustees of UK pension funds, including USS, over the past two decades are UK, European and US equities, US and UK bonds, UK property and cash. In recent years interest in alternative assets has increased, and so we have also included this asset class (represented by hedge funds and commodities). We used 11 assets, and these are set out in the upper section of Table 1. Monthly data on these assets was collected for the period April 1993 to March 2011 (216 observations) corresponding to the triennial actuarial valuations of USS, and we used monthly returns. Although all its liabilities are denominated in sterling, USS has substantial investments in non-sterling assets (about £15 billion in 2013). Until 2006 USS did not hedge any of this foreign exchange risk, but thereafter USS hedged all its foreign exchange risk. Therefore asset returns are adjusted onto a sterling basis for April 2006 onwards. Finally, we ignore the administrative expenses and transaction costs of the scheme.

Table 1: Dataset for Asset Classes and Factors

Type	Source
Asset Classes	
UK Equities	FTSE All Share Total Return
EU Equities	MSCI Europe ex. UK Total Return
US Equities	S&P500 Total Return
10 year UK Bonds	10-year UK Gov. Yields
20 year UK Bonds	20-year UK Gov. Yields
10 year US Bonds	10-year US Gov. Yields
20 year US Bonds	20-year US Gov. Yields
Hedge Funds	HFRI Hedge Fund Index
Commodities	S&P GSCI Total Return
UK Property	IPD Index Total Return
Cash	UK 3 Month Treasury Bills
Factors	
Global Equities	MSCI World Total Return
20 year UK Bonds	20-year Gov. Yields
UK Expected Inflation	UK 10-year Implied Inflation
UK Avg. Earnings	UK Avg. Earnings Index

Its maximin objective means that robust optimization will tend to perform better than the other techniques when the market falls; while the USS benchmark with its high equity investment will tend to perform better in a rising market. Figure 1 shows that the data period (1993-2011) covers a wide range of economic conditions, with three strong upward trends in the UK equity market, and two substantial falls. Therefore the results are not due to testing the ALM models on a falling market, which would favour robust optimization and penalize USS. Indeed, over the entire data period, the index rose by more than 400%.

Figure 1: FTSE All Share Total Return Index 1993-2011

4.2 Liabilities

The liabilities were split into three groups - active members, deferred pensioners and pensioners. Changes in their value are driven by changes in four main factors - long-term interest rates, expected salary growth, expected inflation and longevity expectations. The actuarial equations in Board and Sutcliffe (2007) were used to compute the monthly returns for each of the three types of liability (see Appendix D). This was done using monthly 20-year UK government bond yields[4], and the monthly index of UK 10-year implied inflation. The 20-year government bond yield was used as the discount rate because, while no cash flow forecasts are available, USS is an immature scheme and data on the age distribution of active members, deferreds and pensioners suggests the duration of USS liabilities is over 20 years (USS, 2013). The USS actuary estimates expected salary growth as expected inflation plus one percent, and so monthly changes in expected salary growth were computed in this way. Monthly data on changes in longevity expectations is not available, and so these expectations were held constant throughout each three year period at the value used in the preceding actuarial valuation (see the last two rows of Table 2).

[4] This abstracts from the effects on returns of the low liquidity of pension liabilities and the inflation risk inherent in government bond yields, as these effects tend to cancel out.

The computation of the liabilities also requires a number of parameters - expected age at retirement, life expectancy at retirement, and the average age of actives and deferreds. Although the USS normal retirement age is 65 years, expected retirement ages are earlier. Row 1 of Table 2 shows the expected retirement ages for actives and deferreds used for each triennial USS actuarial valuation, and row 2 has the expected longevity of USS members at the age of 65 (USS Actuarial Valuations). Since the average age of USS members throughout the period was 46 years (HEFCE, 2010), and the average age of USS pensioners was 70 years (USS, 2013), the number of years for which each group was expected to receive a pension are also shown in rows 3 and 4 of Table 2. The USS accrual rate in the final salary section is 1/80th per year.

Table 2: Demographic Data for Actives, Deferreds and Pensioners

Expected Values in Years	1993/6	1996/9	1999/02	2002/5	2005/8	2008/11
Retirement Age	60	60	60	60	60	62
Longevity at 65	20	20	21	21	21	24
Pension Period - Actives & Deferreds	25	25	26	26	26	27
Pension Period – Pensioners	15	15	16	16	16	19

4.3 Constraints

The upper and lower bounds on the asset proportions of the five main asset classes were set so as to rule out extreme and unacceptable asset proportions. This was done with reference to the benchmarks and the associated permitted active positions specified by USS over the data period. The bounds used were: (35% ≤ Equities ≤ 85%); (5% ≤ Fixed Income ≤ 30%); (0% ≤ Alternative Assets ≤ 30%); (2% ≤ Property ≤ 15%): and (0% ≤ Cash ≤ 5%). In addition, the expected return on the asset-liability portfolio was required to be non-negative, and short sales and borrowing were excluded.

Actuarial valuations of USS are carried out every three years, with the oldest available actuarial valuation on 31st March 1993, and the most recent valuation on 31st March 2011. The data is divided into six non-overlapping periods to coincide with these triennial actuarial valuations, as shown in Table 3.

Table 3: Non-Overlapping Three Year Data Periods

Periods (t)	Start	End	Length (p)
Period 1	1993 M4	1996 M3	36
Period 2	1996 M4	1999 M3	36
Period 3	1999 M4	2002 M3	36
Period 4	2002 M4	2005 M3	36
Period 5	2005 M4	2008 M3	36
Period 6	2008 M4	2011 M3	36

To estimate and test the asset allocations we used four out-of-sample non-overlapping windows. The data for the initial six years (periods one and two) was used to compute the optimal robust optimization asset allocation for the subsequent three years (period three). The data period was then rolled forward by 36 months, so that data for periods two and three was now used to compute the optimal asset allocation, which was tested on data for period four, and so on, giving four out-of-sample test periods of 36 months each, providing 144 out-of-sample months.

Each of the three liabilities was treated as a separate risky 'asset class' with 'negative' and fixed weights for each of the six three year periods. We calculated the proportions for each type of pension liability from the triennial actuarial valuations (see Table 4). For the six year estimation periods we used the average of the liability weights for the two 3-year periods concerned. In computing returns on the asset and liability portfolios, the assets were weighted by the funding ratio at the start of the relevant three year period.

Table 4: Proportions of Total Pension Liabilities

Type	Period 1	Period 2	Period 3	Period 4	Period 5	Period 6
Active	59.50%	57.42%	55.23%	57.36%	52.51%	52.20%
Pensioners	36.05%	36.25%	37.89%	35.01%	39.56%	39.90%
Deferred	4.45%	6.33%	6.88%	7.63%	7.93%	7.90%

4.4 Uncertainty Sets

For each investment period, we calculated the parameters involved in the three uncertainty sets, and hence in the final mathematical optimization problem, using a factor model. The returns uncertainty set requires the estimation of 14 mean returns, and for each estimation period the means of the 14 asset and liability returns were used. This was also the case for the

Sharpe and Tint model. For each of the six year estimation periods natural log returns on the 11 assets and three liabilities were separately regressed on the natural log returns of the four factors listed in the lower section of Table 1, together with a constant term[5]. These 14 regressions per estimation period generated 56 estimated coefficients (the factor loadings matrix) and 14 constant terms. In total, the factor loadings uncertainty set requires the estimation of 56 coefficients and 10 covariances, while the Sharpe and Tint model requires 105 elements of its covariance matrix to be estimated. Finally, the disturbances uncertainty set has 14 parameters, and these were estimated using the residuals from the regressions. So overall the Sharpe and Tint model requires the estimation of (105+14) = 119 stochastic parameters per estimation period; while robust optimisation has (56+10+14+14) = 94 stochastic parameters; a reduction of over 20%.

The adjusted R^2 values and significance of these 14 regressions for the entire data set appear in Table 5 (the results for each of the four 72 month estimation periods were broadly similar). This shows that for eight assets, and all the liabilities, an *F*-test on the significance of the equation was significant at the 0.1% level, and the adjusted R^2 was generally high. The 100% adjusted R^2 for 20 year UK bonds is because 20 year UK bonds was one of the four factors included in the factor model. For cash, commodities and UK property the factor model was a poor fit, although for cash one of the factors was significant at the 5% level and another at the 10% level, and one factor was significant at the 10% level for commodities. While there is considerable uncertainty in estimating the input parameters for these three assets, this will be allowed for by robust optimization, but not by the deterministic model.

[5] We experimented with both the identity and number of factors before settling on the four factors listed in Table 1. Increasing the number of factors from four to five would increase the total number of parameters to be estimated by 19. It is helpful if the ratio of the number of factors to the number of assets and liabilities is small. In previous robust optimization studies this ratio is 0.080 and 0.233 (Goldfarb & Iyengar, 2003); 0.125 and 0.238 (Ling & Xu, 2012) and 0.200 (Glasserman & Xu, 2013). With four factors and 14 assets and liabilities we have a ratio of 0.286, which is higher than previous studies. Five factors would increase this ratio to 0.357, which would be appreciably higher than any previous study. Therefore we settled on a parsimonious model of four factors which the R^2 values in Table 5 show do a good job in explaining returns for most of the 14 assets and liabilities.

Table 5: Adjusted R^2 and Significance Levels of the 14 Regression Equations

	1993-2008	
	Adj.R^2 %	***p*-value**
UK Equities	61.01	0.000
EU Equities	79.09	0.000
US Equities	85.63	0.000
10 year UK Bonds	84.52	0.000
20 year UK Bonds	100.00	0.000
10 year US Bonds	20.49	0.000
20 year US Bonds	26.75	0.000
Hedge Funds	58.46	0.000
Commodities	0.98	0.222
UK Property	0.50	0.302
Cash	1.62	0.144
Actives	89.50	0.000
Deferreds	89.49	0.000
Pensioners	90.09	0.000

The equations in Appendix B were then used to compute the three ellipsoidal uncertainty sets. We varied the confidence level (ω) from zero to unity in steps of 0.01 generating 101 solutions, and regressed the logs of the 101 out-of-sample Sharpe ratios (computed using 144 monthly observations) on the logs of ω used in their computation, together with a constant. This regression had a positive slope that was significantly positive at the 3% level, showing that, as the confidence level (ω) is increased, the out-of-sample Sharpe ratio also tends to increase. When ω is zero there is no robust optimization, and all the parameters are treated as deterministic, i.e. the deterministic portfolio benchmark. As ω tends to unity the uncertainty set becomes larger, and the maximin robust optimization solution uses increasingly unfavourable values of the stochastic parameters. Therefore, when ω is close to unity confidence in the robust solution is at its highest, as is the difference between the robust and deterministic solutions. So we decided to examine values of ω close to unity. The value of ω was not set equal to unity because the required confidence level would become infinite. Since the relationship between ω and the Sharpe ratio is not strictly monotonic, we solved the robust optimization model for 21 values of ω between 0.80 to 0.99 and also 0.999, and used the average asset proportions across these portfolios as the robust optimization solution.

The robust optimization ALM model was solved using SeDuMi 1.03 within MATLAB (Sturm, 1999), and this took 0.67 seconds for each value of ω (or 56 seconds for $4(21) = 84$ values of ω) on a laptop computer with a 2.0 GHz processor, 4 GB of RAM and running Windows 7. The modified Sharpe and Tint model was solved using the fmincon function in MATLAB for constrained nonlinear optimization problems.

5 Results

The asset allocations for robust optimization and the three benchmarks for the four out-of-sample periods appear in Table 6. This shows that, while the robust optimization and deterministic solutions are subject to upper and lower bounds on the asset proportions, these constraints are never binding. We examined the 21 individual solutions for the robust optimization and the deterministic solution for each of the four periods, and in none of these 88 solutions were any of these upper and lower bounds binding. However varying these bounds changes the optimal solutions because the robust counterpart is a nonlinear convex optimisation problem, and its optimal solution need not be at a corner point. Therefore the solutions were affected by the upper and lower bounds. The Sharpe and Tint results are constrained in every period by the upper bound on property, the upper bound on alternatives in the last two periods, the lower bound on equities in the middle two periods, and the lower bound on bonds in the last period.

Table 6 shows that while the robust optimization and deterministic portfolios were quite similar, they had some big differences from the USS solutions. For the first three periods USS had over 80% of the assets invested in equities, while the other two methods had just over 50% in equities. USS had no investment in alternative assets until the last period when it jumped to 16%, while the other two methods had about 12% of the assets in alternatives throughout all four periods. The Sharpe and Tint model has some substantial differences from the other three models.

Table 6: Asset Proportions for Robust Optimization (RO), Deterministic Portfolios (DP), Sharpe and Tint (S&T) and USS

	Period 3 - 1999-2002				Period 4 - 2002-2005				Period 5 - 2005-2008				Period 6 - 2008-2011			
	RO	DP	S&T	USS	RO	DP	S&T	USS	RO	DP	S&T	USS	RO	DP	S&T	USS
UK Equities	18.33	18.08	0.00	57.38	18.79	20.34	31.44	38.81	16.89	20.45	0.00	38.36	16.40	17.41	0.00	23.06
EU Equities	14.77	15.17	10.20	11.46	14.24	13.80	0.00	21.50	12.80	5.60	35.00	22.52	13.67	14.32	50.00	18.32
US Equities	15.23	15.42	57.90	11.46	14.75	12.34	3.56	21.50	16.82	12.96	0.00	22.52	17.36	16.30	0.00	18.32
Total Equities	**48.33**	**48.67**	**68.10**	**80.30**	**47.78**	**46.48**	**35.00**	**81.81**	**46.51**	**39.01**	**35.00**	**83.40**	**47.43**	**48.03**	**50.00**	**59.70**
10-year UK Bonds	6.33	6.00	0.00	2.15	6.35	6.61	0.00	2.72	6.15	7.80	0.00	0.22	6.54	5.78	0.00	1.90
20-year UK Bonds	5.90	5.61	13.90	2.15	6.09	5.84	24.45	2.71	6.38	8.13	0.00	0.22	6.92	5.86	0.00	1.90
10-Year US Bonds	5.21	5.36	0.00	2.35	5.36	5.63	0.00	1.33	5.66	6.37	11.18	4.78	5.12	5.46	0.00	4.25
20-year US Bonds	4.22	4.65	0.00	2.35	4.38	4.49	0.00	1.33	4.99	4.46	8.82	4.78	4.62	5.33	5.00	4.25
Total Bonds	**21.66**	**21.62**	**13.90**	**9.00**	**22.18**	**22.57**	**24.45**	**8.09**	**23.18**	**26.76**	**20.00**	**10.00**	**23.20**	**22.42**	**5.00**	**12.30**
Commodities	7.96	8.46	0.00	0.00	7.32	8.66	0.00	0.00	7.47	9.72	30.00	0.00	6.86	8.37	30.00	8.00
Hedge Funds	8.62	8.42	3.00	0.00	9.17	8.82	25.55	0.00	9.41	10.07	0.00	0.00	9.34	8.27	0.00	8.00
Total Alternatives	**16.58**	**16.88**	**3.00**	**0.00**	**16.49**	**17.48**	**25.55**	**0.00**	**16.88**	**19.79**	**30.00**	**0.00**	**16.20**	**16.65**	**30.00**	**16.00**
UK Property	**9.45**	**9.13**	**15.00**	**8.40**	**9.54**	**9.64**	**15.00**	**8.30**	**9.61**	**10.54**	**15.00**	**3.10**	**9.37**	**9.09**	**15.00**	**7.00**
Cash	**3.98**	**3.70**	**0.00**	**2.30**	**4.01**	**3.83**	**0.00**	**1.80**	**3.82**	**3.90**	**0.00**	**3.50**	**3.80**	**3.81**	**0.00**	**5.00**

Table 7 compares the results for the four methods over the 144 out-of-sample months. The 144 months comprised the four out-of-sample 36 month periods, with different solutions applying for each 36 month period, and the monthly figures were adjusted to give the annualized figures. The first row of Table 7 shows the average out-of-sample annualized average Sharpe ratios across the 144 months, with robust optimization having the highest Sharpe ratio at 0.0768, and Sharpe and Tint the lowest at −0.0223.

The Sharpe ratio is based on the excess returns and standard deviation of returns of the asset-liability portfolio, and the annualized average out-of-sample values of these two numbers appear in rows 2 and 3 of Table 7. Robust optimization had the largest annualized average excess return at 1.316% and the smallest annualized average standard deviation of returns at 17.14%; while Sharpe and Tint had the smallest annualized average excess return at −0.4476%, and the largest annualized average standard deviation of returns at 20.22%. Therefore robust optimization mean-variance dominates the three benchmarks.

Table 7: Out-of-Sample Performance Measures

Performance Measures	Robust Optimization	Deterministic Portfolios	Sharpe & Tint	USS
1. Annualized Mean Sharpe Ratio	0.0768	0.0665	−0.0223	0.0280
2. Annualized Mean Excess Return	1.316%	1.144%	−0.4476%	0.5368%
3. Annualized Mean SD (Returns)	17.14%	17.19%	20.22%	19.05%
4. Annualized Sortino Ratio	0.1115	0.0964	−0.0312	0.0402
5. Dowd Ratio	0.0135	0.0117	−0.0037	0.0042
6. 2nd Order Stochastic Dominance	1	2	4	3
7. Mean Diversification	0.1136	0.1140	0.3141	0.2544
8. Mean Stability	0.0005	0.0070	0.3203	0.035
9. Mean Funding Ratio	92.78%	92.39%	87.89%	89.98%
10. Mean Contribution Rate	33.97%	34.09%	35.49%	34.82%
11. Cumulative Asset Return	30.70%	26.43%	−11.97%	16.56%

Since the Sharpe ratio uses the standard deviation to measure risk, we used three additional performance measures which do not rely on the standard deviation to provide an alternative perspective. The Sortino ratio is the mean return less the minimum acceptable return (the return on the asset-liability portfolio), divided by the standard deviation of returns computed using only deviations below the minimum acceptable return. In row 4 of Table 7 robust optimization has the best Sortino ratio (0.1115), and Sharpe and Tint has the worst (−0.0312). The Dowd ratio is the mean return less the return on the liabilities (the return on the asset-liability portfolio), divided by the value at risk for a chosen confidence level (deflated by the initial value of the portfolio), Prigent (2007). We used the 5% confidence level for the value at risk, and row 5 of Table shows that the best Dowd ratio is for robust optimization (0.0135), and the worst is for Sharpe and Tint (−0.0037). Row 6 of Table 7 has the ranking of models using second order stochastic dominance, and robust optimization dominates the deterministic model, which dominates USS, which dominates Sharpe and Tint.

Diversification of the asset-only portfolios was measured by the sum of the squared portfolio proportions for each period (Blume and Friend, 1974). For zero diversification the score is one, while for full diversification it is $1/n$ (or 0.091 when $n = 11$). The average results across the four periods in row 7 of Table 7 show that the robust optimization portfolios are the most diversified (0.1136), and the Sharpe and Tint portfolios are the least diversified (0.3141).

The stability of portfolio proportions from one triennial period to the next was measured as the sum of squares of the differences between the portfolio proportion for each asset in adjacent time periods (Goldfarb and Iyengar, 2003). This measure can be viewed as a proxy for transactions costs under the assumption that the cost functions are linear and similar across assets. The average results in row 8 of Table 7 for the three out-of-sample changes in asset proportions show that robust optimization produced the most stable portfolios (0.0005), and Sharpe and Tint the least stable (0.3203). This suggests that robust optimization generates lower transaction costs for rebalancing the portfolio between asset classes than do the three benchmarks.

The highest mean funding ratio across the 144 months is achieved by robust optimization at 92.78%, and that for Sharpe and Tint is the lowest at 87.89% (see row 9 of Table 7). The mean contribution rate was computed using the actuarial formulae in Board and Sutcliffe (2007) with a spread period (M) of ten years (see Appendix A). The number of years accrued by the average member (P) was 18 years, while administrative expenses were set to zero. The term N_AS cancels out with terms in AL_A. The values of the discount rate (d') and salary increase (e) were the average values over the preceding two triennial periods. Row 10 of Table 7 shows that robust optimization gives the lowest mean contribution rate (33.97%), while Sharpe and Tint has the highest (35.49%). Cumulative returns for just the asset portfolio over the 144 out-of-sample months are shown in row 11 of Table 7. Robust optimisation has the highest asset growth at 30.70%, while Sharpe and Tint shows a decline of 11.97%.

6 Conclusions

In this paper we extended the robust mean-variance portfolio framework proposed by Goldfarb and Iyengar (2003) by incorporating the risky pension liabilities as separate risky assets with their own fixed 'negative' weights. This framework uses a factor loadings matrix, rather than a covariance matrix, which reduces the number of parameters to be estimated by over 20%. Robust optimization ensures that the probability of the actual outcome being worse than the optimal robust solution is equal to ω, where $0 < \omega < 1$. With its maximin objective function and confidence level, robust optimization tends to rule out solutions based on favourable errors in estimating the stochastic input parameters, so tackling estimation risk. We modelled extra features of the pension ALM problem by adding additional linear constraints which set upper and lower bounds for each asset sub-class, ruled out short selling and borrowing, and required the expected return on the asset-liability portfolio to be non-negative. Our final formulation is computationally tractable and easily solved, which is not the case for ALM models using stochastic programming or dynamic stochastic control.

Our study is unusual because we disaggregate the pension liabilities into three components (active members, deferred members and pensioners). We are the first to use the Sharpe ratio as the objective of a pension ALM model. Furthermore, we used the model developed by Board and Sutcliffe (2007) to transform the optimal asset allocation to the scheme's projected contribution rate. Finally, we are the first to derive optimal investment policies using the robust optimization ALM framework for a real-world pension scheme - USS. The choice of this scheme has the advantages that, as the sponsors are tax exempt, the tax arbitrage investment of 100% in bonds is irrelevant; and that, since default risk of the sponsors is both very low and independent that of the scheme, there is no need to include the assets and liabilities of the sponsors in the ALM. The performance of the robust optimization framework was benchmarked against that of the same ALM model with deterministic parameters, the Sharpe and Tint model, and USS's actual investments. This analysis allowed for USS switching to fully hedging its foreign exchange risk from April 2006.

Robust optimization mean-variance dominates the three benchmarks, i.e. a higher mean return, together with a lower standard deviation of returns , so that robust optimisation has the highest Sharpe ratio. Robust optimisation also has the highest Sortino and Dowd ratios, the most diversified and stable portfolios, the highest mean funding ratio, the lowest mean contribution rate and the highest cumulative asset return. In addition, using second order stochastic dominance, it is preferable to the other three models. We conclude that robust optimization is a promising technique for solving pension scheme ALMs.

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Appendix A - Contribution Rate

The contribution rate (CR) is given by:-

$$\mathrm{CR} = \mathrm{SCR} + \mathrm{kAL_A}\left(1-\mathrm{FR}\right)/(N_A S) \tag{A.1}$$

Where

$$\mathrm{k} = \frac{1}{\sum_{z=0}^{M-1}\left(1+\mathrm{d}\right)^{-z}} \tag{A.2}$$

$$1+\mathrm{d} = \frac{1+\mathrm{d}'}{1+\mathrm{e}} \tag{A.3}$$

$$\mathrm{SCR} = \frac{\mathrm{AL_A}}{\mathrm{P}\times\mathrm{N_A}\times\mathrm{S}\times a_{\overline{1}|}} + \mathrm{AE} \tag{A.4}$$

d' is the discount rate for liabilities

$a_{\overline{1}|}$ is an annuity to give the present value of earnings by the member over the next year

AE is the administrative expenses of the scheme, expressed as a proportion of the current salaries of the active members.

M is the life in years of a compound interest rate annuity - the spread period

FR denotes the funding ratio

AL_A is the actuarial liability for the active members of the scheme (see equation D.1),

P is the average member's past years of service as at the valuation date,

S is the average member's annual salary at the valuation date,

e is the forecast nominal rate of salary growth per annum between the valuation date and retirement,

N_A is the current number of active members of the scheme.

Appendix B - Uncertainty Sets

Given the following factor model for asset and liability returns over a single period:-

$$\tilde{\mathbf{r}}_{\mathrm{A,L}} = \tilde{\boldsymbol{\mu}}_{\mathrm{A,L}} + \tilde{\mathbf{V}}^{\mathrm{T}}\mathbf{f} + \tilde{\boldsymbol{\varepsilon}}_{\mathrm{A,L}} \tag{B.1}$$

the uncertainty structures for the matrix of uncertain factor coefficients $\left(\tilde{\mathbf{V}}\right)$, the column vector of the random asset and liability mean returns $\left(\tilde{\boldsymbol{\mu}}_{\mathrm{A,L}}\right)$ and the diagonal covariance matrix of the uncertain disturbances $\left(\tilde{\mathbf{D}}\right)$ are described as follows:-

$$S_m = \left\{\tilde{\boldsymbol{\mu}}_{\mathrm{A,L}} : \tilde{\boldsymbol{\mu}}_{\mathrm{A,L}} = \boldsymbol{\mu}_{\mathrm{A,L,0}} + \tilde{\boldsymbol{\xi}},\ \left|\tilde{\xi}_i\right| \le \gamma_i\left(\omega\right),\ i = 1,\ldots,n_{\mathrm{A}} + n_{\mathrm{L}}\right\} \tag{B.2}$$

$$S_\upsilon = \left\{\tilde{\mathbf{V}} : \tilde{\mathbf{V}} = \mathbf{V}_0 + \tilde{\mathbf{W}},\ \left\|\tilde{\mathbf{W}}_i\right\|_g \le \rho_i\left(\omega\right),\ i = 1,\ldots,n_{\mathrm{A}} + n_{\mathrm{L}}\right\} \tag{B.3}$$

$$S_d = \left\{\tilde{\mathbf{D}} : \tilde{\mathbf{D}} = \mathrm{diag}\left(\tilde{\mathbf{d}}\right),\ 0 \le \tilde{\mathrm{d}}_i \le \mathrm{d}_{\mathrm{upper},i},\ i = 1,\ldots,n_{\mathrm{A}} + n_{\mathrm{L}}\right\} \tag{B.4}$$

where $\left\|\mathbf{w}\right\|_g = \sqrt{\mathbf{w}^{\mathrm{T}}\mathbf{G}\mathbf{w}}$ defines the elliptic norm of a column vector $\mathbf{w}$ with respect to a symmetric and positive definite matrix $\mathbf{G}$. The choice of the structure of the uncertainty sets is motivated by the fact that the estimates of the matrix of factor loadings $\left(\mathbf{V}_0\right)$ and the column vector of mean returns $\left(\boldsymbol{\mu}_{\mathrm{A,L,0}}\right)$ are computed using multivariate linear regression. The parameterisation of the uncertainty sets e.g. $\gamma_i\left(\omega\right)$, $\rho_i\left(\omega\right)$ and $\mathrm{d}_{\mathrm{upper},i}$ from market data is justified by Goldfarb and Iyengar (2003).

Appendix C – Conversion to a Second Order Cone Problem

The mean return and variance of the asset-liability portfolio follow the same form as in Goldfarb and Iyengar (2003) and hence are given by the following equations:-

$$\mathrm{E}\left[\mathrm{r}_{\boldsymbol{\Phi}_{\mathrm{A,L}}}\right] = \boldsymbol{\Phi}_{\mathrm{A,L}}^{\mathrm{T}} \tilde{\boldsymbol{\mu}}_{\mathrm{A,L}} \tag{C.1}$$

$$\mathrm{Var}\left[\mathrm{r}_{\boldsymbol{\Phi}_{\mathrm{A,L}}}\right] = \boldsymbol{\Phi}_{\mathrm{A,L}}^{\mathrm{T}} \left(\tilde{\mathbf{V}}^{\mathrm{T}} \mathbf{F} \tilde{\mathbf{V}} + \tilde{\mathbf{D}}\right) \boldsymbol{\Phi}_{\mathrm{A,L}} \tag{C.2}$$

The general form of the robust (worst-case) maximum Sharpe ratio is described by the maximin problem given in section 2 in equation (2). In what follows, we derive the 'worst-case' expressions:

$\underset{\{\tilde{\boldsymbol{\mu}}_{\mathrm{A,L}} \in S_m\}}{min} \left[\tilde{\boldsymbol{\mu}}_{\mathrm{A,L}}^{\mathrm{T}} \boldsymbol{\Phi}_{\mathrm{A,L}}\right]$ (worst-case mean return), $\underset{\{(\tilde{\mathbf{V}} \in S_\upsilon),\}}{max} \left[\boldsymbol{\Phi}_{\mathrm{A,L}}^{\mathrm{T}} \tilde{\mathbf{V}}^{\mathrm{T}} \mathbf{F} \tilde{\mathbf{V}} \boldsymbol{\Phi}_{\mathrm{A,L}}\right]$ (worst-case variance 1), $\underset{\{\tilde{\mathbf{D}} \in S_d\}}{max} \left[\boldsymbol{\Phi}_{\mathrm{A,L}}^{\mathrm{T}} \tilde{\mathbf{D}} \boldsymbol{\Phi}_{\mathrm{A,L}}\right]$ (worst-case variance 2), and hence we show how the maximin problem described in section 2 in equation (2) can be formulated as a computationally tractable and easily solved second order cone program (SOCP).

Worst-Case Mean Return: The mean return is expressed as a homogeneous function in terms of the vector of asset proportions $\left(\boldsymbol{\Phi}_{\mathrm{A}}\right)$ as follows:-

$$\tilde{\boldsymbol{\mu}}_{\mathrm{A,L}}^{\mathrm{T}} \boldsymbol{\Phi}_{\mathrm{A,L}} = \tilde{\boldsymbol{\mu}}_{\mathrm{A}}^{\mathrm{T}} \boldsymbol{\Phi}_{\mathrm{A}} + \tilde{\boldsymbol{\mu}}_{\mathrm{L}}^{\mathrm{T}} \boldsymbol{\Phi}_{\mathrm{L}} = \left[\boldsymbol{\mu}_{\mathrm{A}}^{\mathrm{T}} + \boldsymbol{\mu}_{\mathrm{L}}^{\mathrm{T}} \boldsymbol{\Phi}_{\mathrm{L}} \mathbf{1}^{\mathrm{T}}\right] \boldsymbol{\Phi}_{\mathrm{A}} \tag{C.3}$$

Given that the uncertainty in mean returns is specified by S_m $\left(\tilde{\boldsymbol{\mu}}_{\mathrm{A,L}} \in S_m\right)$ and since $\boldsymbol{\Phi}_{\mathrm{A},i} \geq 0,\ i = 1,\ldots,n_{\mathrm{A}}$, $\boldsymbol{\Phi}_{\mathrm{L},i} < 0,\ i = 1,\ldots,n_{\mathrm{L}}$, it can be easily shown that the worst-case mean return is given by the following expression:-

$$\underset{\{\tilde{\boldsymbol{\mu}}_{\mathrm{A,L}} \in S_m\}}{min} \left[\tilde{\boldsymbol{\mu}}_{\mathrm{A,L}}^{\mathrm{T}} \boldsymbol{\Phi}_{\mathrm{A,L}}\right] = \left[\boldsymbol{\mu}_{\mathrm{A},0}^{\mathrm{T}} - \boldsymbol{\gamma}_{\mathrm{A}}^{\mathrm{T}}(\omega) + \left(\boldsymbol{\mu}_{\mathrm{L},0}^{\mathrm{T}} + \boldsymbol{\gamma}_{\mathrm{L}}^{\mathrm{T}}(\omega)\right) \boldsymbol{\Phi}_{\mathrm{L}} \mathbf{1}^{\mathrm{T}}\right] \boldsymbol{\Phi}_{\mathrm{A}} \tag{C.4}$$

Worst-Case Variance 1: The term $\boldsymbol{\Phi}_{\mathrm{A,L}}^{\mathrm{T}} \tilde{\mathbf{V}}^{\mathrm{T}} \mathbf{F} \tilde{\mathbf{V}} \boldsymbol{\Phi}_{\mathrm{A,L}}$ is converted to a homogeneous function in terms of the vector of assets $\left(\boldsymbol{\Phi}_{\mathrm{A}}\right)$ as follows:-

$$\boldsymbol{\Phi}_{\mathrm{A,L}}^{\mathrm{T}} \tilde{\mathbf{V}}^{T} \mathbf{F} \tilde{\mathbf{V}} \boldsymbol{\Phi}_{\mathrm{A,L}} = \left\|\tilde{\mathbf{V}} \boldsymbol{\Phi}_{\mathrm{A,L}}\right\|_f^2 = \left\|\left(\tilde{\mathbf{V}}^{sub} + \tilde{\mathbf{V}}^{'}\right) \boldsymbol{\Phi}_{\mathrm{A}}\right\|_f^2 = \boldsymbol{\Phi}_{\mathrm{A}}^{\mathrm{T}} \left(\tilde{\mathbf{V}}^{sub} + \tilde{\mathbf{V}}^{'}\right)^{\mathrm{T}} \mathbf{F} \left(\tilde{\mathbf{V}}^{sub} + \tilde{\mathbf{V}}^{'}\right) \boldsymbol{\Phi}_{\mathrm{A}} \tag{C.5}$$

where $\tilde{\mathrm{V}}_{i,j}^{\mathrm{sub}} = \tilde{\mathrm{V}}_{i,j}$ and $\tilde{\mathrm{V}}'_{i,j} = \zeta_i = \sum_{k=1}^{n_{\mathrm{L}}} \Phi_{\mathrm{L},k} \tilde{\mathbf{V}}_{i,n_{\mathrm{A}}+k}$ with dimension $m \times n_{\mathrm{A}}$.

Given that the uncertainty in $\tilde{\mathbf{V}}$ is specified by the uncertainty structure S_υ $\left(\tilde{\mathbf{V}} \in S_\upsilon\right)$, it can be shown that the expression for the worst-case variance 1, $\underset{\left\{\left(\tilde{\mathbf{V}} \in S_\upsilon\right),\right\}}{max} \left[\mathbf{\Phi}_{\mathrm{A,L}}^{\mathrm{T}} \tilde{\mathbf{V}}^{\mathrm{T}} \mathbf{F} \tilde{\mathbf{V}} \mathbf{\Phi}_{\mathrm{A,L}}\right]$, is less than ν if and only if:-

$$\underset{\left\{\mathbf{y}:\|\mathbf{y}\|_g \le r\right\}}{max} \left\|\mathbf{y}_0 + \mathbf{y}\right\|_f^2 \le \nu \tag{C.6}$$

where $\mathbf{y}_0 = \left(\mathbf{V}_0^{sub} + \mathbf{V}_0^{'}\right)\mathbf{\Phi}_A$ and $r = \sum_{i=1}^{n_{\mathrm{A}}} \Phi_{\mathrm{A},i} \left[\rho_i(\omega) - \sum_{j=1}^{n_{\mathrm{L}}} \Phi_{\mathrm{L},j} \rho_{n_A+j}\right]$.

Constraint (C.6) can be reformulated as a set of linear equalities, linear inequalities and second order cone constraints (SOCC) as follows:-

$$\left(\sum_{i=1}^{n_A} \Phi_{\mathrm{A},i} \left[\rho_i(\omega) - \sum_{j=1}^{n_{\mathrm{L}}} \Phi_{\mathrm{L},j} \rho_{n_A+j}\right]; \nu; \mathbf{\Phi}_A\right) \in Def\left(\mathbf{V}_0^{sub} + \mathbf{V}_0^{'}, \mathbf{F}, \mathbf{G}\right) \tag{C.7}$$

where given that the matrices $\mathbf{F}$ and $\mathbf{G}$ are positive definite, $Def\left(\mathbf{V}_0^{sub} + \mathbf{V}_0^{'}, \mathbf{F}, \mathbf{G}\right)$ is defined to represent the set of all column vectors $\left(r; \nu; \mathbf{\Phi}_{\mathrm{A}}\right)$ such that $\left(r, \nu, \left(\mathbf{V}_0^{sub} + \mathbf{V}_0^{'}\right)\mathbf{\Phi}_{\mathrm{A}}\right)$ satisfy a set of linear equalities, linear inequalities and SOCC similar to those described in LEMMA 1 in Goldfarb and Iyengar (2003).

Worst-Case Variance 2: The term $\mathbf{\Phi}_{\mathrm{A,L}}^{\mathrm{T}} \tilde{\mathbf{D}} \mathbf{\Phi}_{\mathrm{A,L}}$ is converted to a homogeneous function in terms of the vector of asset weights $\left(\mathbf{\Phi}_{\mathrm{A}}\right)$ as follows:-

$$\mathbf{\Phi}_{\mathrm{A,L}}^{\mathrm{T}} \tilde{\mathbf{D}} \mathbf{\Phi}_{\mathrm{A,L}} = \mathbf{\Phi}_{\mathrm{A}}^{\mathrm{T}} \left(\tilde{\mathbf{D}}' + \tilde{\mathbf{D}}''\right) \mathbf{\Phi}_{\mathrm{A}} \tag{C.8}$$

where $\tilde{\mathbf{D}}' = \mathrm{diag}\left(\tilde{\mathbf{d}}\right)$ and $\tilde{\mathrm{D}}''_{i,j} = \zeta = \sum_{j=1}^{n_{\mathrm{L}}} \tilde{\mathrm{d}}_{n_{\mathrm{A}}+j} \Phi_{\mathrm{L},j}^2$ with dimension $n_{\mathrm{A}} \times n_{\mathrm{A}}$.

Since $\left(\tilde{\mathbf{D}} \in S_d\right)$, it can be easily shown that the expression for the worst-case variance 2, $\underset{\left\{\tilde{\mathbf{D}} \in S_d\right\}}{max} \left[\mathbf{\Phi}_{\mathrm{A,L}}^{\mathrm{T}} \tilde{\mathbf{D}} \mathbf{\Phi}_{\mathrm{A,L}}\right]$, is given by the following expression:-

$$
\begin{aligned}
\underset{\{\tilde{\mathbf{D}}\in S_d\}}{max}\left[\boldsymbol{\Phi}_{\mathrm{A,L}}^{\mathrm{T}}\tilde{\mathbf{D}}\boldsymbol{\Phi}_{\mathrm{A,L}}\right] &= \underset{\{\tilde{\mathbf{D}}\in S_d\}}{max}\left[\boldsymbol{\Phi}_{\mathrm{A}}^{\mathrm{T}}\left(\tilde{\mathbf{D}}'+\tilde{\mathbf{D}}''\right)\boldsymbol{\Phi}_{\mathrm{A}}\right] \\
&= \boldsymbol{\Phi}_{\mathrm{A}}^{\mathrm{T}}\left(\mathbf{D}'_{upper}+\mathbf{D}''_{upper}\right)\boldsymbol{\Phi}_{\mathrm{A}} \\
&= \left\|\left(\mathbf{D}'_{upper}+\mathbf{D}''_{upper}\right)^{\frac{1}{2}}\boldsymbol{\Phi}_{\mathrm{A}}\right\|^2
\end{aligned}
\tag{C.9}
$$

where $\tilde{\mathbf{D}}'_{upper} = \mathrm{diag}\left(\mathbf{d}_{upper}\right)$ and $\tilde{\mathrm{D}}''_{upper,i,j} = \zeta_{upper} = \sum_{k=1}^{n_{\mathrm{L}}} \mathrm{d}_{upper,n_{\mathrm{A}}+k}\Phi_{\mathrm{L},k}^2$ with dimension $n_{\mathrm{A}} \times n_{\mathrm{A}}$.

In addition, the constraints with certain parameters (non-negativity and constraints on portfolio weights) are as follows:

$$\Phi_{\mathrm{A},i} \geq 0,\ i = 1,\ldots,n_{\mathrm{A}} \tag{C.10}$$

$$\mathbf{A}_U^{\mathrm{T}}\boldsymbol{\Phi}_{\mathrm{A}} - \theta_U \mathbf{1}^{\mathrm{T}}\boldsymbol{\Phi}_{\mathrm{A}} \leq 0 \tag{C.11}$$

$$-\mathbf{A}_L^{\mathrm{T}}\boldsymbol{\Phi}_{\mathrm{A}} + \theta_L \mathbf{1}^{\mathrm{T}}\boldsymbol{\Phi}_{\mathrm{A}} \leq 0 \tag{C.12}$$

where $\mathbf{A}_U$ and $\mathbf{A}_L$ are column vectors with ones and zeros indicating which assets participate in the bound constraints. Noting that the constraints on portfolio weights are homogeneous in terms of the vector of asset weights (affine linear constraints) and bringing together (C.4), (C.7), (C.9), (C.10), (C.11) and (C.12), the robust maximum Sharpe ratio is given by the following easily solved and tractable SOCP:-

$$
\begin{aligned}
&\underset{\boldsymbol{\Phi}_{\mathrm{A}}}{minimize} && \nu+\delta \\
&s.t: && \left\|\begin{bmatrix} 2\left(\mathbf{D}'_{upper}+\mathbf{D}''_{upper}\right)^{\frac{1}{2}}\boldsymbol{\Phi}_{\mathrm{A}} \\ 1-\delta \end{bmatrix}\right\| \leq 1+\delta \\
& && \left(\sum_{i=1}^{n_A}\Phi_{\mathrm{A},i}\left[\rho_i(\omega)-\sum_{j=1}^{n_{\mathrm{L}}}\Phi_{\mathrm{L},j}\rho_{n_{\mathrm{A}}+j}\right];\nu;\boldsymbol{\Phi}_{\mathrm{A}}\right) \in Def\left(\mathbf{V}_0^{sub}+\mathbf{V}_0^{'},\mathbf{F},\mathbf{G}\right) \\
& && \left[\boldsymbol{\mu}_{0,\mathrm{A}}^{\mathrm{T}}-\boldsymbol{\gamma}_{\mathrm{A}}^{\mathrm{T}}(\omega)+\left(\boldsymbol{\mu}_{0,\mathrm{L}}^{\mathrm{T}}+\boldsymbol{\gamma}_{\mathrm{L}}^{\mathrm{T}}(\omega)\right)\boldsymbol{\Phi}_{\mathrm{L}}\mathbf{1}^{\mathrm{T}}\right]\boldsymbol{\Phi}_A \geq 0.001 \\
& && -\left(\mathbf{A}_U^{\mathrm{T}}-\theta_U\mathbf{1}^{\mathrm{T}}\right)\boldsymbol{\Phi}_{\mathrm{A}} \geq 0 \\
& && -\left(-\mathbf{A}_L^{\mathrm{T}}+\theta_L\mathbf{1}^{\mathrm{T}}\right)\boldsymbol{\Phi}_{\mathrm{A}} \geq 0 \\
& && \Phi_{\mathrm{A},i} \geq 0,\ i=1,\ldots,n_{\mathrm{A}}
\end{aligned}
\tag{C.13}
$$

The key issue of the final SOCP is that since (C.3), (C.5) and (C.8) are homogeneous in terms of the column vector of asset proportions $(\boldsymbol{\Phi}_A)$, and hence $(\boldsymbol{\Phi}_A)$ is no longer normalized. This increases the tractability of the problem. Homogenization is maintained even if we add extra linear affine constraints.

Appendix D - Actuarial Liability Models[6]

The actuarial liability for active members is:-

$$AL_A = N_A \times \left(\frac{P \times S}{A}\right) \times \left\{\frac{(1+e)}{(1+h)}\right\}^{R-G} \times \left\{\left[1 - \left(1 + \frac{1+h}{1+p}\right)^{-W}\right] \Big/ \left(\frac{1+h}{1+p}\right)\right\} \tag{D.1}$$

Where A is the accrual rate,

h is the nominal discount rate between now and retirement,

R is the average member's forecast retirement age,

G is the average age of the member at the valuation date,

W is the life expectancy of members at retirement,

p is the rate of growth of the price level,

AL_A, P, e, N_A and S are defined in Appendix A

The final term in equation (D.1) is the capital sum required at time R to purchase an index-linked annuity of £1 per year.

A simple model for the computation of the actuarial liability for pensioners is:-

$$AL_P = N_P \times PEN \times \left\{\left[1 - \left(\frac{1+h}{1+p}\right)^{-q}\right] \Big/ \left(\frac{1+h}{1+p} - 1\right)\right\} \tag{D.2}$$

where AL_P is the actuarial liability for pensioners,

N_P is the current number of pensioners,

PEN is the average current pension;

and the final term is the capital sum required now to purchase an index-linked annuity of £1 per year for the life expectancy, q, of pensioners. Adjustments to this simple model are required for dependents' pensions, death lump sum, etc.

[6] When computing monthly changes in the actuarial liability, the terms A, N_A, N_D, N_P, S, S_D and PEN are constant within each triennial period, while the initial values of AL_A, AL_D and AL_P come from the actuarial valuations. Therefore the values of A, N_A, N_D, N_P, S, S_D and PEN are not required.

A similar expression for the liability of deferred pensioners is:-

$$AL_D = N_D \times \left(\frac{P_D \times S_D}{A}\right) \times \left\{\frac{(1+p)}{(1+h)}\right\}^{R-G} \times \left\{\left[1-\left(\frac{1+h}{1+p}\right)^{-W}\right] \Big/ \left(\frac{1+h}{1+p}-1\right)\right\} \quad \text{(D.3)}$$

where AL_D is the actuarial liability for the deferred pensioners of the scheme,

N_D is the current number of deferred pensioners of the scheme,

S_D is the average deferred pensioners' leaving salary, compounded forwards to the valuation date at the inflation rate (*p*), and

P_D is the average deferred pensioner's past years of service as at the valuation date.

The total actuarial liability (AL_T) is

$$AL_T = AL_A + AL_P + AL_D \quad \text{(D.4)}$$

which is the sum of the actuarial liabilities for every active member, pensioner and deferred pensioner.