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Multivariate Elliptical Truncated Moments

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Abstract

In this study, we derived analytic expressions for the elliptical truncated moment generating function (MGF), the zeroth-, first-, and second-order moments of quadratic forms of the multivariate normal, Student's t, and generalised hyperbolic distributions. The resulting formulae were tested in a numerical application to calculate an analytic expression of the expected shortfall of quadratic portfolios with the benefit that moment based sensitivity measures can be derived from the analytic expression. The convergence rate of the analytic expression is fast – one iteration – for small closed integration domains, and slower for open integration domains when compared to the Monte Carlo integration method. The analytic formulae provide a theoretical framework for calculations in robust estimation, robust regression, outlier detection, design of experiments, and stochastic extensions of deterministic elliptical curves results.

Keywords: Multivariate truncated moments, Quadratic forms, Elliptical Truncation, Tail moments, Parametric distributions, Elliptical functions

The first results on truncated moments were concerned with the linear truncated multivariate normal (MVN) distribution, and were provided by Tallis (1961). Tallis (1963) extended the results of linear truncations to the case of elliptical and radial truncation, and Tallis (1965) built on previous results to calculate the moments of a normal distribution with a plane truncation. Mantegna and Stanley (1994) used a truncated Lévy distribution to create a distribution where the sums have slow convergence towards the normal, providing first- and second-order moments. Masoom and Nadarajah (2007) calculated the truncated moments of a generalised Pareto distribution. Arismendi (2013) generalised the results of Tallis (1961) for higher-order moments, and for other elliptical distributions such as the Student's t and the lognormal distributions, and for a finite mixture of multivariate normal distributions.[1]

In this study, we derived analytical formulae for the calculation of the elliptical truncated moments of the multivariate normal distribution. We calculated an analytical expansion of the elliptical truncated moment generating function (MGF), and then derived this expression for the calculation of the elliptical truncated moments. Previous results on elliptical and radial truncated moments on multivarite normal distributions were provided by Tallis (1963). In this research we used the results of Ruben (1962) to derive the analytical expressions. We then applied the multivariate normal results to derive the multivariate Student's t (MST) and the multivariate generalised hyperbolic (MGH) elliptical truncated moments. Our results can be considered an extension of Ruben's (1962) results for the MST and MGH cases. The importance of elliptical truncated moments' expansions are evident in applications such as the design of experiments (Thompson, 1976; Cameron and Thompson, 1986), robust estimation (Cuesta-Albertos et al., 2008), outlier detection

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[1] Johnson et al. (1994) have a review of truncated moments for different continuous distributions.

(Riani et al., 2009; Cerioli, 2010), robust regression (Torti et al., 2012; Riani et al., 2014), robust detection (Cerioli et al., 2014), risk averse selection (Hanasusanto et al., 2014), and statistical estates' estimation (Shi et al., 2014).

Other fields where results on elliptical truncated moments can be successfully applied are in physics and dynamical systems. Although the results in these areas are generally for deterministic functions, in recent years advances in elliptical curves have attracted the attention of important researchers. Melander et al. (1986), Waltz et al. (1994), and Ngan et al. (1996) are examples of applications where the extension from deterministic to stochastic elliptical functions can benefit from elliptical truncated moments' results.

This paper makes three contributions: First, we calculate an analytic expression for the moment generating function of the elliptical truncated zeroth- (probability), first-, and second-order moments of the MVN, MST, and the MGH distributions. At the time of producing this research, it was the first time that this analytic expression for the moment generating function had been derived. The univariate generalised hyperbolic (UGH) distribution is defined in Barndorff-Nielsen (1977) as a variance-mean mixture of a normal distribution and a generalised inverse Gaussian (GIG) distribution, and its properties and applications are studied further in Barndorff-Nielsen and Blaesild (1981). In Barndorff-Nielsen et al. (1982), the UGH distribution is extended to the MGH case. The MGH distribution was introduced in finance by Eberlein and Keller (1995), Barndorff-Nielsen (1997), and Eberlein et al. (1998). An extensive study of the use of the MGH distribution in finance can be found in Eberlein (2001).

Second, the results provided use and extend the theory of *multivariate truncated moments*, as a generalisation that could be used to complement other calculations in applied fields. For example, the expected shortfall is the first moment of the distribution truncated at the losses greater than the *VaR*; the value of a plain-vanilla option is the first moment of the risk-neutral density truncated at the strike price. This generalisation of the concept of truncated moments allows us to use the results from one area of finance, such as option theory, to others such as risk management, and *vice versa*. The first results on the first two-order moments of the MGH distribution were due to Schmidt et al. (2006), and were then extended to higher-order moments by Scott et al. (2011). Broda (2013) presented some results on truncated moments of the MGH distribution, extending the results of Imhof (1961), based on a numerical method of the inversion of the characteristic function. The results of our research complement Broda (2013), as the analytic expression we provide is based on the results on moments of the GIG distribution, that are functions of Bessel of the first and second kind, for which there exist analytic expressions such as in Mehrem et al. (1991).

Third, as a numerical application we provide an analytic expression for the calculation of the elliptical truncated moments of mixtures of multivariate random variables. Expressions for the expected shortfall in the cases of the MVN, MST, and MGH distributions are provided, complementing the results of Broda (2012)[2] on heavy-tailed distributions. In the case of elliptical distributions, Kamdem (2008) calculated the *VaR* and the expected shortfall of a quadratic portfolio for a mixture of elliptical distributions by an integral equation,[3] and Yueh and Wong (2010) provided analytic expressions for *VaR* and the expected shortfall when the risk factors are normally distributed by means of a Fourier transform. Our results improve on Kamdem (2008) and Yueh and Wong (2010), as we provide an analytic expression which is faster to calculate.

The structure of this paper is as follows: Section 1 develops an analytic expression of the expected shortfall in the case of MVN distributions. Section 2 derives the extension of Section 1, for distributions that are mixture with the normal distribution. In Section 3, the analytic expression for the expected shortfall in the case of MGH distributions is presented. In Section 4, applications for risk measurement and numerical results are presented. Section 5 deals with extreme numerical cases and Section 6 presents our the conclusions.

[2]The results of Broda (2012) are an extension of those of Glasserman et al. (2002), from using the *VaR* to using the expected shortfall as a risk measure.

[3]Kamdem (2008) is an extension of the methodology applied by Kamdem (2005), from linear to quadratic portfolios.

1. Analytic expressions for the MGF of elliptical truncated quadratic forms in MVN distributions

MST distributions can be represented as a scale mixture of the MVN distribution and a gamma distribution; similarly, MGH distributions can be represented as a mean–variance mixture of the MVN and a GIG distribution. An expression for the truncated moments of quadratic forms over MVN distributions is required for the results of Sections 2 and 3. The methodology applied in both cases can be easily be replicated for any mixture that includes the MVN distribution, and a distribution with a known moment generating function, such as the skew-normal, mixture of normal and Q-Gaussian distributions.

We calculate the MGF and the first- and second-order elliptical truncated moments of the MVN distribution case in this section, as the base result for the subsequent MGF and truncated moments calculations.

In Tallis (1963), the arbitrary order moments of the MVN distribution with elliptical truncation are found. However, the formula is for an ellipsoid restriction centred at zero. We extend the Tallis (1963) results for a non-centred ellipsoid restriction:

Proposition 1.1. *Let $X = (X_1, \ldots, X_n)$ have the MVN distribution, with mean vector $\boldsymbol{\mu}_X$ and covariance matrix $\boldsymbol{\Sigma}_X$, $a \in \mathbb{R}$. Define an ellipsoid restriction $C(\mathbf{x}, a) = \{\mathbf{x} \in \mathbb{R}^n : a \leq (\mathbf{x} - \boldsymbol{\mu}_A)'\mathbf{A}(\mathbf{x} - \boldsymbol{\mu}_A)\}$. The truncated MGF of X at the ellipsoid $C(X, a)$ is equal to,*

$$\mathbb{E}\left[\exp(\mathbf{t}X)|C(X,a)\right] = m(\mathbf{t}, C(\mathbf{x},a)) \quad = \quad L^{-1}\exp\left(\mathbf{t}'\boldsymbol{\mu}_X + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}_X\mathbf{t}\right) H_{n;\mathbf{E},\mathbf{b}(t)}(a/p), \tag{1}$$

the truncated zeroth-, first-, and second-order moment of X at the ellipsoid $C(X, a)$ is equal to,

$$\begin{aligned} \mathbb{P}\left[X|C(X,a)\right] &= m_0(C(\mathbf{x},a)) = L = H_{n;\mathbf{E},\mathbf{b}_0}(a/p), \qquad (2)\\ \mathbb{E}\left[X|C(X,a)\right] = m_1(C(\mathbf{x},a)) &= \mathbb{E}\left[X|a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] \\ &= \boldsymbol{\mu}_X + L^{-1}\sum_{i=0}^{\infty} G_{n+2i}(a/p)c_{i;[\partial\mathbf{t};\mathbf{0}]}, \qquad (3)\\ \mathbb{E}\left[XX'|C(X,a)\right] = m_2(C(\mathbf{x}),a) &= \boldsymbol{\mu}_X\boldsymbol{\mu}_X' + \boldsymbol{\Sigma}_X + L^{-1}\boldsymbol{\mu}_X\left(\sum_{i=0}^{\infty} G_{n+2i}(a/p)c_{i;[\partial\mathbf{t};\mathbf{0}]}\right)' + \\ & L^{-1}\left(\sum_{i=0}^{\infty} G_{n+2i}(a/p)c_{i;[\partial\mathbf{t};\mathbf{0}]}\right)\boldsymbol{\mu}_X' + L^{-1}\sum_{i=0}^{\infty} G_{n+2i}(a/p)c_{i;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}]}, \qquad (4)\end{aligned}$$

where,

$$H_{n;\mathbf{E},\mathbf{b}(t)}(s) = \sum_{i=0}^{\infty} c_i G_{n+2i}(s), \tag{5}$$

where the diagonal matrix $\mathbf{E} = diag(e_1, \ldots, e_n)$ has the eigenvalues of $\mathbf{PAP}$ with $PP' = \Sigma_X$, and vector $\mathbf{b}(\mathbf{t}) = (b_1(t), \ldots, b_n(t))$ is defined as,

$$\mathbf{b}(\mathbf{t}) = \mathbf{K}^{-1}\left(\boldsymbol{\Sigma}_X^{-1/2}(\boldsymbol{\mu}_A - \boldsymbol{\mu}_X) - \boldsymbol{\Sigma}_X^{1/2}\mathbf{t}\right) = \mathbf{K}^{-1}\mathbf{P}^{-1}\left(\boldsymbol{\mu}_A - \boldsymbol{\mu}_X - \mathbf{t}'\boldsymbol{\Sigma}_X\right),$$

with $\mathbf{K}$ a matrix with the eigenvectors of the orthogonal decomposition of $\boldsymbol{\Sigma}_X^{1/2}\mathbf{A}\boldsymbol{\Sigma}_X^{1/2}$, vector $\mathbf{b}_0 = \{b_{1;0}, \ldots, b_{n;0}\}$ is equal to the vector $\mathbf{b}(\mathbf{t})$ evaluated at $\mathbf{t} = 0$, the coefficients c_i are defined through a recursive equation,

$$\begin{aligned} c_0 &= \exp\left(-\frac{1}{2}\mathbf{b}(t)'\mathbf{b}(t)\right)\prod_{j=1}^{n}(p/e_j)^{1/2}, \qquad (6)\\ c_i &= (2i)^{-1}\sum_{k=0}^{i-1} d_{i-k}c_k, \ \forall i \geq 1, \qquad (7)\end{aligned}$$

and coefficients are equal,

$$d_i \quad = \quad \sum_{j=1}^{n} (1 - p/e_j)^i + ip \sum_{j=1}^{n} \left(b_j(t)^2/e_j\right)(1 - p/e_j)^{i-1}, \tag{8}$$

for $j \in \{1, \dots, n\}$, *coefficients* $c_{i;0}$ *are equal to* c_i *substituting* $\mathbf{b}(\mathbf{t})$ *by* $\mathbf{b}_0$,

$$c_{i;0} \quad \equiv \quad [c_i|_{\mathbf{t}=\mathbf{0}}\,, \tag{9}$$

$G_{n+2i}(s) = 1 - F_{n+2i}(s)$, $F_{n+2i}(s)$ *is the distribution of a central chi-squared with* $n+2i$ *degrees of freedom,*[4] *and the term* $c_{i;[\partial\mathbf{t};\mathbf{0}]}$ *refers to a vector of the partial derivatives of the coefficient* c_i, *where the component* j*-th is* $\left[\frac{\partial c_i}{\partial t_j}\right|_{t=0}$, *then,*

$$c_{i;[\partial\mathbf{t};\mathbf{0}]} \quad \equiv \quad \left[\frac{\partial c_i}{\partial \mathbf{t}}\right|_{\mathbf{t}=\mathbf{0}},$$

$$c_{i;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}]} \quad \equiv \quad \left[\frac{\partial^2 c_i}{\partial \mathbf{t}\partial \mathbf{t}}\right|_{\mathbf{t}=\mathbf{0}}.$$

PROOF. The density of X is,

$$\phi_n(\mathbf{x}, \boldsymbol{\mu}_X, \boldsymbol{\Sigma}_X) = (2\pi)^{-n/2}|\boldsymbol{\Sigma}_X|^{-1/2} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_X)'\boldsymbol{\Sigma}_X^{-1}(\mathbf{x} - \boldsymbol{\mu}_X)\right). \tag{12}$$

To calculate (3) and (4), we use the moment generating function approach of Tallis (1963). Define the abbreviated integral operator as,

$$\int_{a_1}^{\infty} \dots \int_{a_n}^{\infty} (\cdot)dx_1 \dots dx_n = {}^{(n)}\!\!\int_{a_s}^{\infty} (\cdot)d\mathbf{x}, \tag{13}$$

Using the definition in (12), the truncated MGF of X can be calculated as,

$$\begin{aligned} \mathbb{E}[\exp(\mathbf{t}'X)|a \leq (X - \boldsymbol{\mu}_A)'\mathbf{A}(X - \boldsymbol{\mu}_A)] \quad &= \quad m(\mathbf{t}, C(\mathbf{x}, a)) \\ = L^{-1}(2\pi)^{-n/2}|\boldsymbol{\Sigma}_X|^{-1/2}\, {}^{(n)}\!\!\int_{C(\mathbf{x},a)} &\exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_X)'\boldsymbol{\Sigma}_X^{-1}(\mathbf{x} - \boldsymbol{\mu}_X) + \mathbf{t}'\mathbf{x}\right)d\mathbf{x}, \end{aligned} \tag{14}$$

where,

$$L = (2\pi)^{-n/2}|\boldsymbol{\Sigma}|^{-1/2}\, {}^{(n)}\!\!\int_{C(\mathbf{x},a)} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu}_X)'\boldsymbol{\Sigma}_X^{-1}(\mathbf{x} - \boldsymbol{\mu}_X)\right)d\mathbf{x}.$$

Let $\mathbf{y} = \mathbf{x} - \boldsymbol{\mu}_X - \mathbf{t}'\boldsymbol{\Sigma}_X$, then (14) becomes,

$$m(\mathbf{t}, C(\mathbf{x}, a)) \quad = \quad L^{-1}(2\pi)^{-n/2}|\boldsymbol{\Sigma}_X|^{-1/2} \exp\left(\mathbf{t}'\boldsymbol{\mu}_X + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}_X\mathbf{t}\right) {}^{(n)}\!\!\int_{C(\mathbf{y},a)} \exp\left(-\frac{1}{2}\mathbf{y}'\boldsymbol{\Sigma}_X^{-1}\mathbf{y}\right)d\mathbf{y} \tag{15}$$

[4] The central chi-squared cumulative density function with ν degrees of freedom is,

$$F_\nu(x) = \frac{\gamma\,(\nu/2, x/2)}{\Gamma\,(\nu/2)},$$

where $\Gamma\,(x)$ is the gamma function, $\gamma\,(x, y)$ is the lower-incomplete gamma function,

$$\Gamma\,(x) \quad = \quad \int_0^{\infty} t^{x-1} \exp(-t)dt, \tag{10}$$

$$\gamma\,(x, y) \quad = \quad \int_0^{y} t^{x-1} \exp(-t)dt. \tag{11}$$

where, $C(\mathbf{y},a)=\{\mathbf{y}\in\mathbb{R}^n : a\leq(\mathbf{y}-(\boldsymbol{\mu}_A-\boldsymbol{\mu}_X-\mathbf{t}'\boldsymbol{\Sigma}_X))'\,\mathbf{A}\,(\mathbf{y}-(\boldsymbol{\mu}_A-\boldsymbol{\mu}_X-\mathbf{t}'\boldsymbol{\Sigma}_X))\}$

The distribution in (15) is from the MVN, with ellipsoid restriction $C(\mathbf{y},a)$. Applying an orthogonal decomposition $\mathbf{E}=\mathbf{K}'\mathbf{P}'\mathbf{A}\mathbf{P}\mathbf{K}$, where $\mathbf{P}\mathbf{P}'=\boldsymbol{\Sigma}_X$, and $\mathbf{K}$ the orthogonal matrix with the eigenvectors of $\mathbf{P}'\mathbf{A}\mathbf{P}$, setting $\mathbf{b}(\mathbf{t})=(b_1(t),\ldots,b_n(t))$ such that,

$$\mathbf{b}(\mathbf{t})=\mathbf{K}^{-1}\mathbf{P}^{-1}\left(\boldsymbol{\mu}_A-\boldsymbol{\mu}_X-\mathbf{t}'\boldsymbol{\Sigma}_X\right), \tag{16}$$

and defining $\mathbf{z}=\mathbf{K}^{-1}\mathbf{P}^{-1}\mathbf{y}$, the distribution (15) is transformed in,

$$m(\mathbf{t},C(\mathbf{x},a)) \;=\; L^{-1}(2\pi)^{-n/2}\exp\left(\mathbf{t}'\boldsymbol{\mu}_X+\frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}_X\mathbf{t}\right)\,{}^{(n)}\!\!\int_{C(\mathbf{z},a)}\exp\left(-\frac{1}{2}\mathbf{z}'\mathbf{z}\right)d\mathbf{z}, \tag{17}$$

where,

$$C(\mathbf{z},a)=a\leq(\mathbf{z}-\mathbf{b}(\mathbf{t}))'\,\mathbf{E}\,(\mathbf{z}-\mathbf{b}(\mathbf{t}))\,,$$

with $\mathbf{E}$ a diagonal matrix with the eigenvalues of $\mathbf{PAP}$, and diagonal components $e_i, 1\leq i\leq n$. From Ruben (1962), the distribution in (17) can be expressed as a series expansion of central chi-squared random variables,

$$\begin{aligned}m(\mathbf{t},C(\mathbf{x},a)) &= L^{-1}\exp\left(\mathbf{t}'\boldsymbol{\mu}_X+\frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}_X\mathbf{t}\right)\sum_{i=0}^{\infty}G_{n+2i}(a/p)c_i,\\ &= L^{-1}\exp\left(\mathbf{t}'\boldsymbol{\mu}_X+\frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}_X\mathbf{t}\right)H_{n;\mathbf{E},\mathbf{b}(\mathbf{t})}(a/p),\end{aligned} \tag{18}$$

where p is an arbitrary positive constant.[5] Set $\mathbf{t}=\mathbf{0}$ in (31) and define,

$$\mathbf{b}_0=[\mathbf{b}(\mathbf{t})|_{\mathbf{t}=\mathbf{0}}=(b_{1;0},\ldots,b_{n;0})=\mathbf{K}^{-1}\mathbf{P}^{-1}(\boldsymbol{\mu}_A-\boldsymbol{\mu}_X), \tag{19}$$

we derive the value of L, the zeroth-order moment,

$$L \;=\; H_{n;\mathbf{E},\mathbf{b}_0}(a/p),$$

where,

$$H_{n;\mathbf{E},\mathbf{b}_0}(a/p)=\sum_{i=0}^{\infty}G_{n+2i}(a/p)c_{i;0}, \tag{20}$$

and $c_{i;0}, d_{i;0}$ are defined by (9), and (6), (7), and (8) substituting $\mathbf{b}(\mathbf{t})$ by $\mathbf{b}_0$. First-order moments (3) are calculated deriving the moment generating function,

$$\begin{aligned}\mathbb{E}\left[X|a\leq(X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] &= m_1(C(\mathbf{x},a))=\left.\frac{\partial m(\mathbf{t})}{\partial\mathbf{t}}\right|_{\mathbf{t}=\mathbf{0}}\\ &= \left[\frac{\partial}{\partial\mathbf{t}}\left(\exp\left(\mathbf{t}'\boldsymbol{\mu}_X+\frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}_X\mathbf{t}\right)\frac{H_{n;\mathbf{E},\mathbf{b}(\mathbf{t})}(a/p)}{H_{n;\mathbf{E},\mathbf{b}_0}(a/p)}\right)\right|_{\mathbf{t}=\mathbf{0}}\\ &= \boldsymbol{\mu}_X+\frac{1}{H_{n;\mathbf{E},\mathbf{b}_0}(a/p)}\left[\left.\frac{\partial H_{n;\mathbf{E},\mathbf{b}(\mathbf{t})}(a/p)}{\partial\mathbf{t}}\right|_{\mathbf{t}=\mathbf{0}}\right.\\ &= \boldsymbol{\mu}_X+\frac{1}{H_{n;\mathbf{E},\mathbf{b}_0}(a/p)}\sum_{i=0}^{\infty}G_{n+2i}(a/p)\left[\left.\frac{\partial c_i}{\partial\mathbf{t}}\right|_{\mathbf{t}=\mathbf{0}}\right..\end{aligned}$$

[5] In Ruben (1962) an upper bound for p is derived, $p<\min(e_i), 1\leq i\leq n$. In Genz and Bretz (2009), they referenced an algorithm of Sheil and O'Muircheartaigh (1977) where a series of values for p are tested, finding that $p=29/32\min(e_i)$ had an optimal balance between the speed of the algorithm and the convergence.

The partial derivative of coefficients is expressed as a recursive equation. Define,

$$\begin{aligned} c_{i;[\partial\mathbf{t};\mathbf{0}]} &\equiv \left[\frac{\partial c_i}{\partial\mathbf{t}}\right|_{\mathbf{t}=\mathbf{0}}, \\ c_{i;0} &\equiv \left[c_i\right|_{\mathbf{t}=\mathbf{0}}. \end{aligned}$$

The term $c_{i;[\partial\mathbf{t};\mathbf{0}]}$ refers to a vector where the component j-th is $\left[\frac{\partial c_i}{\partial t_j}\right|_{t=0}$. We derive,

$$\begin{aligned} \left[\frac{\partial}{\partial t_k} b_j\right|_{t_k=0} &= -\mathbf{K}^{-1}_{j,:}\mathbf{P}^{-1}\mathbf{\Sigma}_{X;(:,k)}, \\ \left[\frac{\partial}{\partial t_k} b_j^2\right|_{t_k=0} &= -2b_{j;0}\mathbf{K}^{-1}_{j,:}\mathbf{P}^{-1}\mathbf{\Sigma}_{X;(:,k)}, \end{aligned}$$

where $\mathbf{K}^{-1}_{i,:}$ is the row i-th of the inverse of the eigenvector matrix $\mathbf{K}^{-1}$. Then,

$$c_{0;[\partial\mathbf{t};\mathbf{0}]} = \exp\left(-\frac{1}{2}\mathbf{b}_0'\mathbf{b}_0\right)\left(\mathbf{b}_0'\mathbf{K}^{-1}\mathbf{P}^{-1}\mathbf{\Sigma}_X\right)'\prod_{j=1}^{n}(p/e_j)^{1/2}, \tag{21}$$

$$c_{i;[\partial\mathbf{t};\mathbf{0}]} = (2i)^{-1}\left(\sum_{k=0}^{i-1} d_{i-k;[\partial\mathbf{t};\mathbf{0}]}c_{k;0} + \sum_{k=0}^{i-1} d_{i-k;0}c_{k;[\partial\mathbf{t};\mathbf{0}]}\right), i \geq 1, \tag{22}$$

$$d_{i;[\partial\mathbf{t};\mathbf{0}]} = -2ip\left((\boldsymbol{\lambda}\odot\mathbf{K}^{-1})\mathbf{P}^{-1}\mathbf{\Sigma}_X\right), \tag{23}$$

where $\odot$ is the element to element matrix multiplication, $\boldsymbol{\lambda}(i) = \{\lambda_1(i), \ldots, \lambda_n(i)\}, \lambda_j(i) = \frac{(1-p/e_j)^{i-1}}{e_j}b_{j;0}$, $c_{0;[\partial\mathbf{t};\mathbf{0}]}$ is a vector of dimension n that has as component j-th,

$$\exp\left(-\frac{1}{2}\mathbf{b}_0'\mathbf{b}_0\right)\left(\mathbf{b}_0'\mathbf{K}^{-1}\mathbf{P}^{-1}\mathbf{\Sigma}_{X,(:,j)}\right)\prod_{i=1}^{n}(p/e_i)^{1/2},$$

and $d_{i;[\partial\mathbf{t};\mathbf{0}]}$ is a vector of dimension n that has as component j-th,

$$-2ip(\boldsymbol{\lambda}\odot\mathbf{K}^{-1})\mathbf{P}^{-1}\mathbf{\Sigma}_{X;(:,j)},$$

with $\mathbf{\Sigma}_{X;(j,:)}$ and $\mathbf{P}_{:,j}$ the j-th column of $\mathbf{\Sigma}_X$ and $\mathbf{P}$.

Second-order moments in (4) are calculated deriving $m(\mathbf{t})$ once more,

$$\begin{aligned} \mathbb{E}\left[XX'|a \leq (X-\boldsymbol{\mu})'\mathbf{\Sigma}_X^{-1}(X-\boldsymbol{\mu})\right] &= m_2(C(\mathbf{x},a)) = \left.\frac{\partial^2 m(\mathbf{t})}{\partial\mathbf{t}\partial\mathbf{t}}\right|_{\mathbf{t}=\mathbf{0}} \\ &= \left[\frac{\partial^2}{\partial\mathbf{t}\partial\mathbf{t}}\left(\exp\left(\mathbf{t}'\boldsymbol{\mu}_X + \frac{1}{2}\mathbf{t}'\mathbf{\Sigma}_X\mathbf{t}\right)\frac{H_{n;\mathbf{E},\mathbf{b}}(a/p)}{H_{n;\mathbf{E},\mathbf{b}_0}}(a/p)\right)\right|_{\mathbf{t}=\mathbf{0}} \\ &= \boldsymbol{\mu}_X\boldsymbol{\mu}_X' + \mathbf{\Sigma}_X + \boldsymbol{\mu}_X\frac{1}{H_{n;\mathbf{E},\mathbf{b}_0}(a/p)}\left[\frac{\partial H_{n;\mathbf{E},\mathbf{b}}(a/p)}{\partial\mathbf{t}}\right|_{\mathbf{t}=\mathbf{0}}' + \\ &\quad + \frac{1}{H_{n;\mathbf{E},\mathbf{b}_0}(a/p)}\left[\frac{\partial H_{n;\mathbf{E},\mathbf{b}}(a/p)}{\partial\mathbf{t}}\right|_{\mathbf{t}=\mathbf{0}}\boldsymbol{\mu}_X' + \frac{1}{H_{n;\mathbf{E},\mathbf{b}_0}(a/p)}\left[\frac{\partial^2 H_{n;\mathbf{E},\mathbf{b}}(a/p)}{\partial\mathbf{t}\partial\mathbf{t}}\right|_{\mathbf{t}=\mathbf{0}} \\ &= \boldsymbol{\mu}_X\boldsymbol{\mu}_X' + \mathbf{\Sigma}_X + \boldsymbol{\mu}_X\left(\frac{\sum_{i=0}^{\infty}G_{n+2i}(a/p)c_{i;[\partial\mathbf{t};\mathbf{0}]}}{H_{n;\mathbf{E},\mathbf{b}_0}(a/p)}\right)' + \left(\frac{\sum_{i=0}^{\infty}G_{n+2i}(a/p)c_{i;[\partial\mathbf{t};\mathbf{0}]}}{H_{n;\mathbf{E},\mathbf{b}_0}(a/p)}\right)\boldsymbol{\mu}_X' + \\ &\quad + \frac{\sum_{i=0}^{\infty}G_{n+2i}(a/p)c_{i;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}]}}{H_{n;\mathbf{E},\mathbf{b}_0}(a/p)}, \end{aligned} \tag{24}$$

where $c_{i;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}]} \equiv \left[\frac{\partial^2 c_i}{\partial\mathbf{t}\partial\mathbf{t}}\Big|_{\mathbf{t}=\mathbf{0}}\right]$, and $c_{i;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}]}$ is a matrix with (j,k)-th component $\left[\frac{\partial^2 c_i}{\partial t_j\partial t_k}\Big|_{t_j,t_k=0}\right]$.

We calculate $c_{i;[\partial\mathbf{t}\partial\mathbf{t}=\mathbf{0}]}$. Let,

$$\left[\frac{\partial^2 b_j^2}{\partial t_i \partial t_k}\Big|_{t_i,t_k=\mathbf{0}}\right] = 2\left(\mathbf{K}_{j,:}^{-1}\mathbf{P}^{-1}\boldsymbol{\Sigma}_{X;(:,i)}\right)\left(\mathbf{K}_{j,:}^{-1}\mathbf{P}^{-1}\boldsymbol{\Sigma}_{X;(:,k)}\right),$$

where $\boldsymbol{\Sigma}_{X;(:,k)}$ is the k-th column of the matrix $\boldsymbol{\Sigma}_X$. Then,

$$\begin{aligned} c_{0;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}]} &= \exp\left(-\frac{1}{2}\mathbf{b}_0'\mathbf{b}_0\right)\prod_{j=1}^{n}(p/e_j)^{1/2}\Big(\left(\mathbf{b}_0'\mathbf{K}^{-1}\mathbf{P}^{-1}\boldsymbol{\Sigma}_X\right)\left(\mathbf{b}_0'\mathbf{K}^{-1}\mathbf{P}^{-1}\boldsymbol{\Sigma}_X\right)' - \\ &\qquad \left(\mathbf{K}^{-1}\mathbf{P}^{-1}\boldsymbol{\Sigma}_X\right)\left(\mathbf{K}^{-1}\mathbf{P}^{-1}\boldsymbol{\Sigma}_X\right)'\Big), \qquad (25) \\ c_{i;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}]} &= (2i)^{-1}\sum_{k=0}^{i-1}\Big(d_{i-k;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}]}c_{k;0} + d_{i-k;[\partial\mathbf{t};\mathbf{0}]}c'_{k;[\partial\mathbf{t};\mathbf{0}]} + c_{k;[\partial\mathbf{t};\mathbf{0}]}d'_{i-k;[\partial\mathbf{t};\mathbf{0}]} + \\ &\qquad + d_{i-k;0}c_{k;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}]}\Big),\ i \geq 1, \qquad (26) \\ d_{i;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}]} &= 2ip\left((\boldsymbol{\Lambda}\odot\mathbf{K}^{-1})\mathbf{P}^{-1}\boldsymbol{\Sigma}_X\right)'\left(\mathbf{K}^{-1}\mathbf{P}^{-1}\boldsymbol{\Sigma}_X\right), \qquad (27) \end{aligned}$$

where $\boldsymbol{\Lambda} = (\boldsymbol{\lambda},\ldots,\boldsymbol{\lambda})$ is a $n \times n$ matrix with $\boldsymbol{\lambda}$ on each column. The terms in (25), (26), and (27) are matrices. Substituting the definition of L in (24) yields the result.

Example 1. *Let X have a bivariate normal distribution with $\boldsymbol{\mu}_X = (0.10, 0.12)'$ and,*

$$\boldsymbol{\Sigma}_X = \begin{pmatrix} 0.3 & 0.1 \\ 0.1 & 0.2 \end{pmatrix},$$

Let $a = 0.3$, $\mathbf{a} = (0.1, 0.2)'$, and,

$$\mathbf{A} = \begin{pmatrix} 0.2 & 0.05 \\ 0.05 & 0.05 \end{pmatrix},$$

Define a truncation ellipsoid $C(\mathbf{x}, a)$ with $\boldsymbol{\mu}_A = -\frac{1}{2}\mathbf{a}'\mathbf{A}^{-1}$, such that $C(\mathbf{x}, a) = \{\mathbf{x} \in \mathbb{R}^n : a \leq (\mathbf{x} - \boldsymbol{\mu}_A)'\mathbf{A}(\mathbf{x} - \boldsymbol{\mu}_A)\}$. To test the results and the propositions in this study we developed numerical algorithms in `MATLAB`. Applying Proposition 1.1, set $N = 250$, the zeroth-, first-, and second-order moments of X truncated at $C(\mathbf{x}, a)$ are,

$$m_0(C(\mathbf{x},a)) = 0.4556, \qquad m_1(C(\mathbf{x},a)) = \begin{pmatrix} 0.4081 \\ 0.4343 \end{pmatrix}, \qquad m_2(C(\mathbf{x},a)) = \begin{pmatrix} 0.4940 & 0.2113 \\ 0.2113 & 0.3224 \end{pmatrix}.$$

To compare the results, we generated a Monte Carlo simulation with the following results,

$$m_0(C(\mathbf{x},a)) = 0.4555, \qquad m_1(C(\mathbf{x},a)) = \begin{pmatrix} 0.4079 \\ 0.4343 \end{pmatrix}, \qquad m_2(C(\mathbf{x},a)) = \begin{pmatrix} 0.4936 & 0.2111 \\ 0.2111 & 0.3222 \end{pmatrix},$$

with the Monte Carlo simulation of X truncated at $C(\mathbf{x}, a)$ having a standard error of $\left(0.1810 \times 10^{-3}, 0.1156 \times 10^{-3}\right)'$.

2. Analytic expressions for the MGF of an elliptical truncated multivariate Student's t distribution

In this section we use and extend the results of Section 1, to calculate the MGF for the case of quadratic forms where the random variable follows the multivariate Student's t distribution.

The methodology we apply is based on the fact that the Student's t distribution is derived from a scale mixture of the gamma and the normal distributions, and the methodology can be extended to cases where the distributions are the result of scale, mean, and variance mixtures of the normal distribution with other distributions with known truncated moments.

Let $X = (X_1, \ldots, X_n)$ have a multivariate Student's t density,

$$f(\mathbf{x}, \boldsymbol{\mu}_X, \boldsymbol{\Sigma}_X, \nu) = \frac{\Gamma((\nu+n)/2)}{(\pi\nu)^{\nu/2}\Gamma(\nu/2)|\boldsymbol{\Sigma}_X|^{1/2}} \left(1 + \frac{1}{\nu}(\mathbf{x}-\boldsymbol{\mu}_X)'\boldsymbol{\Sigma}_X^{-1}(\mathbf{x}-\boldsymbol{\mu}_X)\right)^{-(\nu+n)/2}, \tag{28}$$

where $\pi = \Gamma(1/2)$, and let η be a random variable with a gamma distribution, shape $\alpha = \nu/2$, scale $\beta = 2/\nu$, and pdf,

$$f_\eta(x; \alpha, \beta) = \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} \exp(-x/\beta). \tag{29}$$

Define $\mathbb{E}_\eta[\cdot]$ as the expected value with respect to η. Assume without loss of generality (w.l.o.g.) that $\boldsymbol{\mu}_X = 0$ for the calculation of the moments, as for cases where $\boldsymbol{\mu}_X \neq \mathbf{0}$, we can provide results with a change of variable $Y = X - \boldsymbol{\mu}_X$, $\mu_Y = \mathbf{0}$ and translate the moment results of Y into X.

We calculate the truncated MGF and truncated moments.

Proposition 2.1. *Let Z have the MVN distribution with pdf (12). Let η have a gamma distribution with pdf (29). Define X as the scale mixture,*

$$X = (X_1, \ldots, X_n) = \eta^{-1/2} Z.$$

Then X has a multivariate standard Student's t-distribution with ν degrees of freedom.

Proof. Let us define the scale mixture of a gamma distribution and an MVN distribution as $X_i = \eta^{-1/2} Z_i$. Then the distribution of X conditional on η is,

$$f_{\eta^{-1/2}Z|\eta} = (2\pi)^{-n/2}|\boldsymbol{\Sigma}_X|^{-1/2} \exp\left(-\frac{1}{2\eta^{-1}}\mathbf{x}'\boldsymbol{\Sigma}_X^{-1}\mathbf{x}\right)\eta^{n/2}. \tag{30}$$

But (30) is the pdf of $N(\mathbf{0}, \eta^{-1}\boldsymbol{\Sigma}_X)$. We have that $f_{\eta^{-1/2}Z} = f_{\eta^{-1/2}Z|\eta} f_\eta$ with f_η equal to (29) with parameters $\alpha = \nu/2$, $\beta = 2/\nu$. Hence,

$$\begin{aligned} f_{\eta^{-1/2}Z} &= (2\pi)^{-n/2}|\boldsymbol{\Sigma}_X|^{-1/2} \int_0^\infty \exp\left(-\frac{1}{2\eta^{-1}}\mathbf{x}'\boldsymbol{\Sigma}_X^{-1}\mathbf{x}\right)\eta^{n/2} \frac{1}{(2/\nu)^{\nu/2}\Gamma(\nu/2)}\eta^{\nu/2-1}\exp\left(-\frac{\eta}{2/\nu}\right) d\eta, \\ &= \frac{\Gamma((\nu+n)/2)}{(\Gamma(1/2)/\nu)^{\nu/2}\Gamma(\nu/2)|\boldsymbol{\Sigma}_X|^{1/2}} \left(1 + \frac{1}{\nu}\mathbf{x}'\boldsymbol{\Sigma}_X^{-1}\mathbf{x}\right)^{-(\nu+n)/2}, \end{aligned}$$

which is the density function of the MST distribution.

Proposition 2.2. *Let X have an MST distribution as in (28). Define $C(\mathbf{x}, a) = \{\mathbf{x} \in \mathbb{R}^n : a \leq (\mathbf{x} - \boldsymbol{\mu}_A)'\mathbf{A}(\mathbf{x}-\boldsymbol{\mu}_A)\}$, then X has an approximate elliptical truncated MGF over the region $C(X, a)$ denoted by,*

$$m(\mathbf{t}, C(\mathbf{x}, a)) = \mathbb{E}_\eta\left[L_\eta^{-1}\exp\left(\eta^{-1/2}\mathbf{t}'\boldsymbol{\mu}_Z + \eta^{-1}\frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}_Z\mathbf{t}\right) H_{n;\mathbf{E},\mathbf{b}(\eta^{-1/2}\mathbf{t})}(\eta a/p)\right], \tag{31}$$

where,

$$\begin{aligned} H_{n;\mathbf{E},\mathbf{b}(\eta^{-1/2}\mathbf{t})}(\eta a/p) &= \sum_{i=0}^\infty G_{n+2i}(\eta a/p) c_{i;\eta^{-1/2}\mathbf{t}}, \\ L_\eta = H_{n;\mathbf{E},\mathbf{b}_{0;\eta}}(\eta a/p) &= \sum_{i=0}^\infty G_{n+2i}(\eta a/p) c_{i;0;\eta}, \end{aligned} \tag{32}$$

with coefficients $c_{i;\eta^{-1/2}\mathbf{t}}$ equal to coefficients c_i as in (6) and (7) substituting $\mathbf{b}(\mathbf{t})$ by $\mathbf{b}(\eta^{-1/2}\mathbf{t})$ defined by,

$$\mathbf{b}(\eta^{-1/2}\mathbf{t}) = \mathbf{K}^{-1}\mathbf{P}^{-1}\left(\eta^{1/2}\boldsymbol{\mu}_A - \boldsymbol{\mu}_X - \eta^{-1/2}\mathbf{t}'\boldsymbol{\Sigma}_X\right),$$

and coefficients $c_{i;0;\eta}$ equal to c_i as in (6) and (7) substituting $\mathbf{b}(t)$ by $\mathbf{b}_{0;\eta}$ defined by,

$$\mathbf{b}_{0;\eta} \quad = \quad (b_{1;0;\eta}, \ldots, b_{n;0;\eta}) = \mathbf{K}^{-1}\mathbf{P}^{-1}(\eta^{1/2}\boldsymbol{\mu}_A - \boldsymbol{\mu}_X). \tag{33}$$

Assume without loss of generality that $\boldsymbol{\mu}_X = \mathbf{0}$. The elliptical truncated zeroth-, first- and second-order moments of X at $C(X,a)$ are,

$$\mathbb{P}\left[X | a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] \quad = \quad L = \sum_{i=0}^{\infty}\sum_{j=0}^{i} \zeta_{j,i} c_{i_{a_j};0}, \tag{34}$$

$$\mathbb{E}\left[X | a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] \quad = \quad L^{-1}\sum_{i=0}^{\infty}\sum_{j=0}^{i} \zeta_{j+\frac{1}{2},i} c_{i_{a_j};[\partial\mathbf{t};0]}, \tag{35}$$

$$\mathbb{E}\left[XX' | a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] \quad = \quad L^{-1}\sum_{i=0}^{\infty}\sum_{j=0}^{i} \zeta_{j+1,i} c_{i_{a_j};[\partial\mathbf{t}\partial\mathbf{t};0]}, \tag{36}$$

where,

$$\begin{aligned}\zeta_{j,i} \quad = \quad & (1+\mathbf{b}_0'\mathbf{b}_0/\nu)^{-\nu/2-j}\left(\frac{2}{\nu}\right)^{j}\frac{\Gamma\left(\frac{n+2i+2j+\nu}{2}\right)}{\Gamma\left(\frac{n+2i}{2}\right)\Gamma\left(\nu/2\right)}B\left(\frac{\nu}{2}+j,\frac{n+2i}{2}\right)\times \\ & I_{\nu\left(1+\mathbf{b}_0'\mathbf{b}_0/\nu\right)/(\nu+\mathbf{b}_0'\mathbf{b}_0+a/p)}\left(\frac{\nu}{2}+j,\frac{n+2i}{2}\right),\end{aligned} \tag{37}$$

and $c_{i_{a_j};0}$ are numerical coefficients calculated by solving the recurrence (7) for the MST distribution case.

Proof. The truncated MGF of X at the ellipsoid $C(X,a)$ can be approximated as,

$$\mathbb{E}\left[\exp(\mathbf{t}X) | C(X,a)\right] = m(\mathbf{t}, C(X,a)) \quad = \quad \mathbb{E}_\eta[\mathbb{E}[\exp(\mathbf{t}\eta^{-1/2}Z)|\eta, C(X,a)]], \tag{38}$$

where the condition $C(X,a)$ can be transformed as,

$$\begin{aligned} C(X,a) = a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A) \quad &= \quad a \leq \eta^{-1}(Z-\eta^{1/2}\boldsymbol{\mu}_A)'\mathbf{A}(Z-\eta^{1/2}\boldsymbol{\mu}_A) \\ &= \quad \eta a \leq (Z-\eta^{1/2}\boldsymbol{\mu}_A)'\mathbf{A}(Z-\eta^{1/2}\boldsymbol{\mu}_A), \\ &= \quad C(\eta^{-1/2}Z, a). \end{aligned}$$

But using the results of Section 1 the internal expression of (38), $\mathbb{E}[\exp(\mathbf{t}\eta^{-1/2}Z)|\eta, C(\eta^{-1/2}Z,a)]$, is the MGF of an MVN distribution, and it can be calculated as,

$$\mathbb{E}[\exp(\mathbf{t}\eta^{-1/2}Z)|\eta, C(\eta^{-1/2}Z,a)] \quad = \quad L_\eta^{-1}\exp\left(\eta^{-1/2}\mathbf{t}'\boldsymbol{\mu}_Z + \eta^{-1}\frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}_Z\mathbf{t}\right)H_{n;\mathbf{E},\mathbf{b}(\eta^{-1/2}\mathbf{t})}(\eta a/p). \tag{39}$$

Then, the MGF expression (38) can be approximated by,

$$m(\mathbf{t}, C(X,a)) = \mathbb{E}_\eta\left[L_\eta^{-1}\exp\left(\eta^{-1/2}\mathbf{t}'\boldsymbol{\mu}_Z + \eta^{-1}\frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}_Z\mathbf{t}\right)H_{n;\mathbf{E},\mathbf{b}(\eta^{-1/2}t)}(\eta a/p)\right]. \tag{40}$$

If we set $\mathbf{t} = \mathbf{0}$ in (40), we have as a result that,

$$1 = \mathbb{E}_\eta\left[L_\eta^{-1}H_{n;\mathbf{E},\mathbf{b}_{0;\eta}}(\eta a/p)\right], \tag{41}$$

and equality in (32) will hold if $\mathbb{E}_\eta[\cdot]$ exists. In the case $a-\boldsymbol{\mu}'_A\boldsymbol{\mu}_A \le 0$, the integral of the MGF approximation in (40) and the integral (41) are the integrals of a continuous density function over a compact set and we have a convergence of the integral. Otherwise, if the MGF of the non-truncated variable is not convergent the MGF over the truncated region can be not convergent.

The truncated zeroth-order moment (probability) of X at the ellipsoid $C(X,a)$ is,

$$L = \mathbb{E}_\eta[L_\eta]. \tag{42}$$

To calculate (42), we need to develop the series (32),

$$\begin{aligned}
\mathbb{E}_\eta\left[\sum_{i=0}^{\infty} G_{n+2i}(\eta a/p)c_{i;0;\eta}\right] &= \mathbb{E}_\eta\left[G_n(\eta a/p)c_{0;0;\eta}\right] + \mathbb{E}_\eta\left[G_{n+2}(\eta a/p)c_{1;0;\eta}\right] + \mathbb{E}_\eta\left[G_{n+4}(\eta a/p)c_{2;0;\eta}\right] + \dots,\\
&= \mathbb{E}_\eta\left[G_n(\eta a/p)\exp\left(-\frac{1}{2}\mathbf{b}'_{0;\eta}\mathbf{b}_{0;\eta}\right)\prod_{j=1}^{n}(p/e_j)^{1/2}\right] + \\
&\quad \mathbb{E}_\eta\left[G_{n+2}(\eta a/p)2^{-1}\exp\left(-\frac{1}{2}\mathbf{b}'_{0;\eta}\mathbf{b}_{0;\eta}\right)\prod_{j=1}^{n}(p/e_j)^{1/2}d_{1;0;\eta}\right] + \\
&\quad \mathbb{E}_\eta\left[G_{n+4}(\eta a/p)4^{-1}\exp\left(-\frac{1}{2}\mathbf{b}'_{0;\eta}\mathbf{b}_{0;\eta}\right)\prod_{j=1}^{n}(p/e_j)^{1/2}\left(d_{2;0;\eta} + 2^{-1}d^2_{1;0;\eta}\right)\right] + \dots,
\end{aligned} \tag{43}$$

and,

$$d_{i;0;\eta} = \sum_{j=1}^{n}(1-p/e_i)^i + ip\sum_{j=1}^{n}\left(b^2_{j;0;\eta}/e_i\right)(1-p/e_j)^{i-1}.$$

Terms $\mathbf{b}_{0;\eta}$,d_i and $G_{n+2i}(\eta a/p)$ in (43) contain η, then for calculating the $\mathbb{E}_\eta[\cdot]$ we use a series expansion approach. First, considering $\boldsymbol{\mu}_X = \mathbf{0}$ the following factors can be applied to terms dependent on η,

$$\begin{aligned}
\mathbf{b}'_{0;\eta}\mathbf{b}_{0;\eta} &= \eta\mathbf{b}'_0\mathbf{b}_0,\\
d_{i;0;\eta} &= \sum_{j=1}^{n}(1-p/e_i)^i + ip\eta\sum_{j=1}^{n}\left(b^2_{j;0}/e_i\right)(1-p/e_j)^{i-1},\\
&= dA_i + dB_i\eta,
\end{aligned} \tag{44}$$

with $dA_i = \sum_{j=1}^{n}(1-p/e_i)^i$, $dB_i = ip\sum_{j=1}^{n}\left(b^2_{j;0}/e_i\right)(1-p/e_j)^{i-1}$. After applying (44) to the recurrence of $c_{i;0;\eta}$ as in (6), (7), and (8) with the corresponding change (33), lead us to denote the coefficients $c_{i;0;\eta}$ around two terms, $\exp\left(-\frac{1}{2}\eta\mathbf{b}'_0\mathbf{b}_0\right)$ and η as,

$$c_{i;0;\eta} = \exp\left(-\frac{1}{2}\eta\mathbf{b}'_0\mathbf{b}_0\right)\left(c_{i_{a_0};0} + c_{i_{a_1};0}\eta + c_{i_{a_2};0}\eta^2 + \dots + c_{i_{a_i};0}\eta^i\right), \tag{45}$$

with $c_{i_{a_j};0}, i, j \ge 0$ that are coefficients not dependent on η. The value of the coefficients $c_{i_{a_j};0}$ are found by equating (7) with (45) and substituting $\mathbf{b}(t)$ by $\mathbf{b}_{0;\eta}$, considering the relation derived in (44). Hence, the terms in the series (43) can be denoted by,

$$\begin{aligned}
\mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)c_{i;0;\eta}\right] &= \mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\exp\left(-\frac{1}{2}\eta\mathbf{b}'_0\mathbf{b}_0\right)\right]c_{i_{a_0};0} + \\
&\quad \mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta\exp\left(-\frac{1}{2}\eta\mathbf{b}'_0\mathbf{b}_0\right)\right]c_{i_{a_1};0} + \dots + \\
&\quad \mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta^i\exp\left(-\frac{1}{2}\eta\mathbf{b}'_0\mathbf{b}_0\right)\right]c_{i_{a_i};0}.
\end{aligned} \tag{46}$$

We calculate $\mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta^j \exp\left(-\frac{1}{2}\eta\mathbf{b}_0'\mathbf{b}_0\right)\right]$ for $j\in\{1,\dots,i\}$ applying the definitions of the chi-squared distribution and $\mathbb{E}_\eta$,

$$\mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta^j \exp\left(-\frac{1}{2}\eta\mathbf{b}_0'\mathbf{b}_0\right)\right] = \int_0^\infty\int_{\eta a/p}^\infty \frac{x^{(n+2i)/2-1}\exp\left(-\frac{1}{2}x\right)}{2^{(n+2i)/2}\Gamma\left(\frac{n+2i}{2}\right)} \times \frac{\eta^{\nu/2-1}\exp\left(-\frac{\eta}{2/\nu}\right)}{(2/\nu)^{\nu/2}\Gamma(\nu/2)}\times \eta^j \exp\left(-\frac{1}{2}\eta\mathbf{b}_0'\mathbf{b}_0\right)dx d\eta, \quad (47)$$

where $\nu/2$ and $2/\nu$ are the shape and scale parameters of the η variable. Apply the change of variable $\eta y = x$ and (47) becomes,

$$\begin{aligned} &= \int_0^\infty\int_{a/p}^\infty \frac{(\eta y)^{(n+2i)/2-1}\exp\left(-\frac{1}{2}\eta y\right)}{2^{(n+2i)/2}\Gamma\left(\frac{n+2i}{2}\right)} \times \frac{\eta^{\nu/2+j}\exp\left(-\frac{\eta}{2/\nu}\right)\exp\left(-\frac{1}{2}\eta\mathbf{b}_0'\mathbf{b}_0\right)}{(2/\nu)^{\nu/2}\Gamma(\nu/2)}dy d\eta, \\ &= \int_0^\infty\int_{a/p}^\infty \frac{\eta^{(n+2i+2j+\nu)/2-1}y^{(n+2i)/2-1}\exp\left(-\frac{1}{2}\eta(y+\nu+\mathbf{b}_0'\mathbf{b}_0)\right)}{2^{(n+2i)/2}\Gamma\left(\frac{n+2i}{2}\right)(2/\nu)^{\nu/2}\Gamma(\nu/2)}dy d\eta. \end{aligned}$$

Now apply the change of variables $u^2 = y$ and later $w = \frac{1}{2}\eta\left(u^2+\nu+\mathbf{b}_0'\mathbf{b}_0\right)$, hence,

$$= \int_0^\infty\int_{(a/p)^{1/2}}^\infty \frac{(2w/(u^2+\nu+\mathbf{b}_0'\mathbf{b}_0))^{(n+2i+2j+\nu)/2-1}u^{n+2i-2}\exp(-w)}{2^{(n+2i)/2}\Gamma\left(\frac{n+2i}{2}\right)(2/\nu)^{\nu/2}\Gamma(\nu/2)} \times 2(u^2+\nu+\mathbf{b}_0'\mathbf{b}_0)^{-1}(2u)du dw.$$

Applying Fubini and by definition of the function $\Gamma(\cdot)$, we have,

$$\begin{aligned} &= \int_{(a/p)^{1/2}}^\infty\int_0^\infty \frac{2^{j+1}\nu^{\nu/2}w^{(n+2i+2j+\nu)/2-1}(u^2+\nu+\mathbf{b}_0'\mathbf{b}_0)^{-(n+2i+2j+\nu)/2}u^{n+2i-1}\exp(-w)}{\Gamma\left(\frac{n+2i}{2}\right)\Gamma(\nu/2)}dw du \\ &= \int_{(a/p)^{1/2}}^\infty \frac{2^{j+1}\nu^{\nu/2}\Gamma\left(\frac{n+2i+2j+\nu}{2}\right)(u^2+\nu)^{-(n+2i+2j+\nu)/2}u^{n+2i-1}}{\Gamma\left(\frac{n+2i}{2}\right)\Gamma(\nu/2)}du, \end{aligned}$$

then,

$$= 2^{j+1}\frac{\Gamma\left(\frac{n+2i+2j+\nu}{2}\right)}{\Gamma\left(\frac{n+2i}{2}\right)\Gamma(\nu/2)}\int_{(a/p)^{1/2}}^\infty \nu^{\nu/2}(u^2+\nu+\mathbf{b}_0'\mathbf{b}_0)^{-(n+2i+2j+\nu)/2}u^{n+2i-1}du. \quad (48)$$

To solve (48), we apply the change of variable $s = \nu\left(u^2+\nu+\mathbf{b}_0'\mathbf{b}_0\right)^{-1}(1+\mathbf{b}_0'\mathbf{b}_0/\nu)$, then we have,

$$\begin{aligned} &= 2^{j+1}\frac{\Gamma\left(\frac{n+2i+2j+\nu}{2}\right)}{\Gamma\left(\frac{n+2i}{2}\right)\Gamma(\nu/2)}\int_{\nu\left(1+\mathbf{b}_0'\mathbf{b}_0/\nu\right)/(\nu+\mathbf{b}_0'\mathbf{b}_0+a/p)}^0 \nu^{-(n+2i+2j)/2}\left(s\left(1+\mathbf{b}_0'\mathbf{b}_0/\nu\right)^{-1}\right)^{(n+2i+2j+\nu)/2}\nu^{(n+2i-2)/2}\times \\ &\left(s\left(1+\mathbf{b}_0'\mathbf{b}_0/\nu\right)^{-1}\right)^{-(n+2i-2)/2}(1-s)^{(n+2i-2)/2}\left(-\frac{1}{2}\right)\nu\left(s\left(1+\mathbf{b}_0'\mathbf{b}_0/\nu\right)^{-1}\right)^{-2}\left(1+\mathbf{b}_0'\mathbf{b}_0/\nu\right)^{-1}ds \\ &= (1+\mathbf{b}_0'\mathbf{b}_0/\nu)^{-\nu/2-j}\left(\frac{2}{\nu}\right)^j\frac{\Gamma\left(\frac{n+2i+2j+\nu}{2}\right)}{\Gamma\left(\frac{n+2i}{2}\right)\Gamma(\nu/2)}\int_0^{\nu\left(1+\mathbf{b}_0'\mathbf{b}_0/\nu\right)/(\nu+\mathbf{b}_0'\mathbf{b}_0+a/p)} s^{\nu/2+j-1}(1-s)^{(n+2i-2)/2}ds. \end{aligned}$$

Therefore,

$$\mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta^j \exp\left(-\frac{1}{2}\eta\mathbf{b}_0'\mathbf{b}_0\right)\right] = (1+\mathbf{b}_0'\mathbf{b}_0/\nu)^{-\nu/2-j}\left(\frac{2}{\nu}\right)^j\frac{\Gamma\left(\frac{n+2i+2j+\nu}{2}\right)}{\Gamma\left(\frac{n+2i}{2}\right)\Gamma(\nu/2)}\times B\left(\frac{\nu}{2}+j,\frac{n+2i}{2}\right)I_{\nu\left(1+\mathbf{b}_0'\mathbf{b}_0/\nu\right)/(\nu+\mathbf{b}_0'\mathbf{b}_0+a/p)}\left(\frac{\nu}{2}+j,\frac{n+2i}{2}\right), \quad (49)$$

where $B(y,z)$ is the beta function and $I_x(y,z)$ is the lower incomplete beta function. Having (43), (45), (46), and (49) the solution for (42) is derived.

The elliptical truncated first-order moments of X are calculated,

$$\begin{aligned}\mathbb{E}\left[X|a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] &= \mathbb{E}_\eta[\mathbb{E}[\eta^{-1/2}Z|\eta, a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)]] \\ &= \mathbb{E}_\eta[\mathbb{E}[\eta^{-1/2}Z|\eta, a \leq \eta^{-1}(Z-\eta^{1/2}\boldsymbol{\mu}_A)'\mathbf{A}(Z-\eta^{1/2}\boldsymbol{\mu}_A)]] \\ &= \mathbb{E}_\eta[\eta^{-1/2}\mathbb{E}[Z|\eta, \eta a \leq (Z-\eta^{1/2}\boldsymbol{\mu}_A)'\mathbf{A}(Z-\eta^{1/2}\boldsymbol{\mu}_A)]]. \qquad (50)\end{aligned}$$

The internal expression,

$$\mathbb{E}[Z|\eta, \eta a \leq (Z-\eta^{1/2}\boldsymbol{\mu}_A)'\mathbf{A}(Z-\eta^{1/2}\boldsymbol{\mu}_A)],$$

of (50) is the first-order expected value of a normal distribution truncated with the ellipsoid $\eta a \leq (Z-\eta^{1/2}\boldsymbol{\mu}_A)'\mathbf{A}(Z-\eta^{1/2}\boldsymbol{\mu}_A)$ similar to (39), then the results of Section 1 can be used,

$$\mathbb{E}[Z|\eta, \eta a \leq (Z-\eta^{1/2}\boldsymbol{\mu}_A)'\mathbf{A}(Z-\eta^{1/2}\boldsymbol{\mu}_A)] = \boldsymbol{\mu}_Z + L_\eta^{-1}\sum_{i=0}^{\infty} G_{n+2i}(\eta a/p)c_{i;[\partial\mathbf{t};\mathbf{0}];\eta},$$

where $c_{i;[\partial\mathbf{t};\mathbf{0}];\eta}$ are equal to $c_{i;[\partial\mathbf{t};\mathbf{0}]}$ as in (22) substituting $\mathbf{b}_0$ by $\mathbf{b}_{0;\eta}$. Then, the elliptical truncated first-order moment is,

$$\begin{aligned}\mathbb{E}\left[X|a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] &= \mathbb{E}_\eta\left[\eta^{-1/2}\boldsymbol{\mu}_Z + \eta^{-1/2}L_\eta^{-1}\sum_{i=0}^{\infty} G_{n+2i}(\eta a/p)c_{i;[\partial\mathbf{t};\mathbf{0}];\eta}\right] \\ &= \boldsymbol{\mu}_X + \sum_{i=0}^{\infty}\mathbb{E}_\eta\left[\eta^{-1/2}L_\eta^{-1}G_{n+2i}(\eta a/p)c_{i;[\partial\mathbf{t};\mathbf{0}];\eta}\right].\end{aligned}$$

Using the definition and properties of the conditional expectation we have that,

$$\mathbb{E}\left[X|a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] = \boldsymbol{\mu}_X + L^{-1}\sum_{i=0}^{\infty}\mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta^{-1/2}c_{i;[\partial\mathbf{t};\mathbf{0}];\eta}\right]. \qquad (51)$$

The expected value in (51), is solved introducing $\eta^{-1/2}$ inside the coefficients $c_{i;[\partial\mathbf{t};\mathbf{0}];\eta}$ and applying the decomposition in (45), (46) for $c_{i;[\partial\mathbf{t};\mathbf{0}];\eta}$,

$$c_{i;[\partial\mathbf{t};\mathbf{0}];\eta}\eta^{-1/2} = \exp\left(-\frac{1}{2}\eta\mathbf{b}_0'\mathbf{b}_0\right)\left(c_{i_{a_0};[\partial\mathbf{t};0]}\eta^{-1/2} + c_{i_{a_1};[\partial\mathbf{t};0]}\eta^{1/2} + c_{i_{a_2};[\partial\mathbf{t};0]}\eta^{3/2} + \cdots + c_{i_{a_i};[\partial\mathbf{t};0]}\eta^{i-1/2}\right), \qquad (52)$$

and,

$$\begin{aligned}\mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta^{-1/2}c_{i;[\partial\mathbf{t};\mathbf{0}];\eta}\right] &= \mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta^{-1/2}\exp\left(-\frac{1}{2}\eta\mathbf{b}_0'\mathbf{b}_0\right)\right]c_{i_{a_0};[\partial\mathbf{t};0]} + \\ &\quad \mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta^{1/2}\exp\left(-\frac{1}{2}\eta\mathbf{b}_0'\mathbf{b}_0\right)\right]c_{i_{a_1};[\partial\mathbf{t};0]} + \cdots + \\ &\quad \mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta^{i-1/2}\exp\left(-\frac{1}{2}\eta\mathbf{b}_0'\mathbf{b}_0\right)\right]c_{i_{a_i};[\partial\mathbf{t};0]}. \qquad (53)\end{aligned}$$

The solutions to the internal integrals in (53) are solved as in (49),

$$\begin{aligned}\mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta^{j-1/2}\exp\left(-\frac{1}{2}\eta\mathbf{b}_0'\mathbf{b}_0\right)\right] &= (1+\mathbf{b}_0'\mathbf{b}_0/\nu)^{-(\nu+1)/2-j}\left(\frac{2}{\nu}\right)^{j+\frac{1}{2}}\frac{\Gamma\left(\frac{n+2i+2j+1+\nu}{2}\right)}{\Gamma\left(\frac{n+2i}{2}\right)\Gamma\left(\nu/2\right)}\times \\ & B\left(\frac{\nu+1}{2}+j, \frac{n+2i}{2}\right)I_{\nu\left(1+\mathbf{b}_0'\mathbf{b}_0/\nu\right)/(\nu+\mathbf{b}_0'\mathbf{b}_0+a/p)}\left(\frac{\nu+1}{2}+j, \frac{n+2i}{2}\right), \qquad (54)\end{aligned}$$

therefore, having (51), (52), (53), and (54), yields the result on truncated first-order moments.

The truncated second-order moment of X over $C(X,a)$ is calculated,

$$
\begin{aligned}
\mathbb{E}\left[XX'|a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] &= m_2(C(\mathbf{x},a)) \\
&= \mathbb{E}_\eta[E[\eta^{-1}ZZ'|\eta, a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)]] \\
&= \mathbb{E}_\eta[\mathbb{E}[\eta^{-1}ZZ'|\eta, a \leq \eta^{-1}(Z-\eta^{1/2}\boldsymbol{\mu}_A)'\mathbf{A}(Z-\eta^{1/2}\boldsymbol{\mu}_A)]] \\
&= \mathbb{E}_\eta[\eta^{-1}\mathbb{E}[ZZ'|\eta, \eta a/p \leq (Z-\eta^{1/2}\boldsymbol{\mu}_A)'\mathbf{A}(Z-\eta^{1/2}\boldsymbol{\mu}_A)]] \\
&= L^{-1}\mathbb{E}_\eta\left[\eta^{-1}\sum_{i=0}^{\infty} G_{n+2i}(\eta a/p)c_{i;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}];\eta}\right] \\
&= L^{-1}\sum_{i=0}^{\infty}\mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta^{-1}c_{i;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}];\eta}\right], \qquad (55)
\end{aligned}
$$

where coefficients $c_{i;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}];z}$ are equal to $c_{i;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}]}$ as in (26) substituting $\mathbf{b}_0$ by $\mathbf{b}_{0;\eta}$. The expected value in (55), is solved by introducing η^{-1} inside the coefficients $c_{i;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}];\eta}$ and applying the decomposition in (45), (46) for $c_{i;[\partial\mathbf{t};\mathbf{0}];\eta}$,

$$
\begin{aligned}
c_{i;[\partial\mathbf{t}\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}];\eta}\eta^{-1} = \exp\left(-\frac{1}{2}\eta\mathbf{b}_0'\mathbf{b}_0\right)\Big(& c_{i_{a_0};[\partial\mathbf{t}\partial\mathbf{t};0]}\eta^{-1} + c_{i_{a_1};[\partial\mathbf{t}\partial\mathbf{t};0]} + c_{i_{a_2};[\partial\mathbf{t}\partial\mathbf{t};0]}\eta^{1} + \cdots + \\
& c_{i_{a_i};[\partial\mathbf{t}\partial\mathbf{t};0]}\eta^{i-1}\Big), \qquad (56)
\end{aligned}
$$

and,

$$
\begin{aligned}
\mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta^{-1}c_{i;[\partial\mathbf{t};\mathbf{0}];\eta}\right] &= \mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta^{-1}\exp\left(-\frac{1}{2}\eta\mathbf{b}_0'\mathbf{b}_0\right)\right]c_{i_{a_0};[\partial\mathbf{t};0]} + \\
&\quad \mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\exp\left(-\frac{1}{2}\eta\mathbf{b}_0'\mathbf{b}_0\right)\right]c_{i_{a_1};[\partial\mathbf{t};0]} + \cdots + \\
&\quad \mathbb{E}_\eta\left[G_{n+2i}(\eta a/p)\eta^{i-1}\exp\left(-\frac{1}{2}\eta\mathbf{b}_0'\mathbf{b}_0\right)\right]c_{i_{a_i};[\partial\mathbf{t};0]}. \qquad (57)
\end{aligned}
$$

The solutions to the internal integrals in (57) are solved as in (49), then having (55), (56), and (57), the result on the truncated second-order moments is derived.

Example 2. *Let X have a Student's t distribution as in (28) with $\nu = 5$, $\boldsymbol{\mu}_X = (0,0)'$ and $\boldsymbol{\Sigma}_X$ defined as in Example 1. Let $a, \mathbf{a}, \mathbf{A}, \boldsymbol{\mu}_A$, and $C(\mathbf{x},a)$ be defined as in Example 1. Applying Proposition 2.2, set $N = 250$, the zeroth-, first-, and second-order moments of X truncated at $C(\mathbf{x},a)$ are,*

$$
m_0(C(\mathbf{x},a)) = 0.4069, \qquad m_1(C(\mathbf{x},a)) = \begin{pmatrix} 0.2527 \\ 0.3400 \end{pmatrix}, \qquad m_2(C(\mathbf{x},a)) = \begin{pmatrix} 0.9734 & 0.3340 \\ 0.3340 & 0.5157 \end{pmatrix}.
$$

To compare the results, we generated a Monte Carlo simulation with the following results,

$$
m_0(C(\mathbf{x},a)) = 0.4075, \qquad m_1(C(\mathbf{x},a)) = \begin{pmatrix} 0.2526 \\ 0.3404 \end{pmatrix}, \qquad m_2(C(\mathbf{x},a)) = \begin{pmatrix} 0.9689 & 0.3325 \\ 0.3328 & 0.5129 \end{pmatrix},
$$

with the Monte Carlo simulation of X truncated at $C(\mathbf{x},a)$ having a standard deviation of $(0.9548, 0.6323)'$ and a standard error of $\left(0.9548 \times 10^{-3}, 0.6323 \times 10^{-3}\right)$.

3. Analytic expressions for the MGF of the elliptical truncated multivariate generalised hyperbolic distribution

In this section, we apply the results of Section 1, and the methodology of Section 2, to calculate the truncated moments of the MGH distribution.

Let $X = (X_1, \ldots, X_n)$ be the multivariate random variable with the MGH distribution, the density of X is,

$$f_X = \frac{\bar{\alpha}^{n/2}(1-\boldsymbol{\beta}'\boldsymbol{\beta})^{\bar{\lambda}/2} K_{\bar{\lambda}-n/2}\left(\bar{\alpha}\sqrt{1+(\mathbf{x}-\boldsymbol{\mu}_X)'\boldsymbol{\Sigma}_X^{-1}(\mathbf{x}-\boldsymbol{\mu}_X)}\right)}{(2\pi)^{n/2}K_{\bar{\lambda}}\left(\bar{\alpha}\sqrt{1-\boldsymbol{\beta}'\boldsymbol{\beta}}\right)(1+(\mathbf{x}-\boldsymbol{\mu}_X)'\boldsymbol{\Sigma}_X^{-1}(\mathbf{x}-\boldsymbol{\mu}_X))^{n/4-\bar{\lambda}/2}} \exp\left(\bar{\alpha}\boldsymbol{\beta}'\boldsymbol{\Sigma}_X^{-1/2}(\mathbf{x}-\boldsymbol{\mu}_X)\right), \tag{58}$$

where $K_x(\cdot)$ is the modified Bessel function of third-kind, $\boldsymbol{\mu} \in \mathbb{R}^n$ is a location parameter, $\boldsymbol{\Sigma}_X \in \mathbb{R}^{n\times n}$ is a positive definite dispersion parameter, $\boldsymbol{\beta} \in \mathbb{R}^n$ is an asymmetry parameter, $\bar{\alpha} \in \mathbb{R}^+$ is a scale parameter, and $\bar{\lambda}$ a parameter used to produce close distributions in the marginals under affine transformations.

Barndorff-Nielsen (1977) defined the generalised hyperbolic distribution, as a mean–variance mixture. As in Schmidt et al. (2006), let $X, Z \in \mathbb{R}^n$ be random variables with Z distributed as a generalised inverse Gaussian,

$$Z \sim GIG(\bar{\lambda}, \bar{\delta}, \sqrt{\bar{\delta}^2(\bar{\alpha}^2 - \boldsymbol{\beta}'\boldsymbol{\Sigma}_X\boldsymbol{\beta})}), \tag{59}$$

and define $W = (W_1, \ldots, W_n) \equiv X|Z$, as a random variable with normal distribution,

$$X|Z \equiv W \sim N\left(\boldsymbol{\mu}_X + z\boldsymbol{\Sigma}_X\boldsymbol{\beta}, z\boldsymbol{\Sigma}_X\right), \tag{60}$$

then the unconditional distribution of X is MGH with density (58). Schmidt et al. (2006) demonstrated the relationship between the parameters of X, W, and Z for X to be MGH and (60) to hold. In order to simplify the calculations, let $\bar{\delta} = |\boldsymbol{\Sigma}_X|^{1/n}$, following Schmidt et al. (2006) relationship (60) is denoted by,

$$X|Z \equiv W \sim N\left(\boldsymbol{\mu}_X + z\Delta_X\boldsymbol{\beta}, z\Delta_X\right), \tag{61}$$

where $\Delta_X = \boldsymbol{\Sigma}_X/|\boldsymbol{\Sigma}_X|^{1/n}$. Assume w.l.o.g. as in Section (2) that $\boldsymbol{\mu}_X = \mathbf{0}$ for calculating the moments. Using (60), we apply the same methodology calculating the truncated moments of the scale mixture of the normal distribution in Section 2.

Proposition 3.1. *Let X be a random vector with MGH distribution (58). Let Z be distributed as a generalised inverse Gaussian as in (59), and W be the conditional distribution of $W \equiv X|Z$. Define $C(\mathbf{x}, a) = \{\mathbf{x} \in \mathbb{R}^n : a \leq (\mathbf{x}-\boldsymbol{\mu}_A)'\mathbf{A}(\mathbf{x}-\boldsymbol{\mu}_A)\}$, then X has an approximate elliptical truncated MGF over the region $C(X, a)$ denoted by,*

$$m(\mathbf{t}, C(X,a)) = \mathbb{E}_z\left[L_z^{-1}\exp\left(\mathbf{t}'\left(z^{1/2}\Delta_X\boldsymbol{\beta}\right) + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}_X\mathbf{t}\right)H_{n;\mathbf{E},\mathbf{b}_z(t)}(z^{-1}a/p)\right],$$

where,

$$\begin{aligned} H_{n;\mathbf{E},\mathbf{b}_z(\mathbf{t})}(z^{-1}a/p) &= \sum_{i=0}^{\infty} G_{n+2i}(z^{-1}a/p)c_{i;\mathbf{t};z}, \\ L_z = H_{n;\mathbf{E},\mathbf{b}_{0;z}}(z^{-1}a/p) &= \sum_{i=0}^{\infty} G_{n+2i}(z^{-1}a/p)c_{i;0;z}, \end{aligned} \tag{62}$$

with coefficients $c_{i;\mathbf{t};z}$ equal to coefficients c_i as in (6) and (7) substituting $\mathbf{b}(\mathbf{t})$ by $\mathbf{b}_z(\mathbf{t})$ defined by,

$$\mathbf{b}_z(\mathbf{t}) = \mathbf{K}^{-1}\mathbf{P}^{-1}\left(z^{-1/2}\boldsymbol{\mu}_A - \left(z^{1/2}\Delta_X\boldsymbol{\beta}\right) - \mathbf{t}'\Delta_X\right),$$

and coefficients $c_{i;0;z}$ equal to c_i as in (6) and (7) substituting $\mathbf{b}(t)$ by $\mathbf{b}_{0;z}$ defined by,

$$\mathbf{b}_{0;z} = (b_{1;0;z}, \ldots, b_{n;0;z}) = \mathbf{K}^{-1}\mathbf{P}^{-1}(z^{-1/2}\boldsymbol{\mu}_A - z^{1/2}\Delta_X\boldsymbol{\beta}). \tag{63}$$

Assume without loss of generality that $\boldsymbol{\mu}_X = \mathbf{0}$. The elliptical truncated zeroth-, first-, and second-order moments are,

$$\mathbb{P}\left[X|a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] = L = \sum_{i=0}^{\infty}\sum_{j=-i}^{i}\zeta_{j,i}c_{i_{a_j};0}, \tag{64}$$

$$\mathbb{E}\left[X|a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] = L^{-1}\left(\boldsymbol{\mu}_W\sum_{i=0}^{\infty}\sum_{j=-i}^{i}\zeta_{j+1,i}c_{i_{a_j};0} + \sum_{i=0}^{\infty}\sum_{j=-i}^{i+1}\zeta_{j,i}c_{i_{a_j};[\partial\mathbf{t};0]}\right), \tag{65}$$

$$\begin{aligned}\mathbb{E}\left[XX'|a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] = L^{-1}\Bigg(&\left(\sum_{i=0}^{\infty}\sum_{j=-i}^{i}\zeta_{j+2,i}c_{i_{a_j};0}\right)\boldsymbol{\mu}_W\boldsymbol{\mu}_W' + \left(\sum_{i=0}^{\infty}\sum_{j=-i}^{i}\zeta_{j+1,i}c_{i_{a_j};0}\right)\boldsymbol{\Sigma}_W + \\ &\boldsymbol{\mu}_W\left(\sum_{i=0}^{\infty}\sum_{j=-i}^{i+1}\zeta_{j+1,i}c_{i_{a_j};[\partial\mathbf{t};0]}\right)' + \left(\sum_{i=0}^{\infty}\sum_{j=-i}^{i+1}\zeta_{j+1,i}c_{i_{a_j};[\partial\mathbf{t};0]}\right)\boldsymbol{\mu}_W' + \sum_{i=0}^{\infty}\sum_{j=-i}^{i+2}\zeta_{j+2,i}c_{i_{a_j};[\partial\mathbf{t}\partial\mathbf{t};0]}\Bigg),\end{aligned} \tag{66}$$

where,

$$\begin{aligned}\zeta_{j,i} = \frac{(\psi/\chi)^{\lambda/2}\exp\left(-\frac{1}{2}B_{0;0}\right)}{2\Gamma\left(\frac{n+2i}{2}\right)K_\lambda(\sqrt{\chi\psi})}&\left(\Gamma\left(\frac{n+2i}{2}\right)\left(\frac{\chi_B}{\psi_B}\right)^{(\lambda+j+1)/2}K_{\lambda+j+1}\left(\sqrt{\chi_B\psi_B}\right) - \right.\\ &\left.\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}(a/p)\right)^{(n+2i)/2+k}}{\prod_{s=0}^{k}\left(\frac{n+2i}{2}+s\right)}\left(\frac{\chi_{Bap}}{\psi_B}\right)^{(\lambda+j-(n+2)/2-k+1)/2}K_{\lambda+j-(n+2i)/2-k+1}\left(\sqrt{\chi_{Bap}\psi_B}\right)\right),\end{aligned} \tag{67}$$

and $c_{i_{a_j};0}$ are numerical coefficients calculated by solving the recurrence (7) for the MGH distribution case.

Proof. Let Z be a random vector with a GIG distribution, with pdf,

$$f_Z = \frac{(\bar{p}/\bar{\delta})^{\bar{\lambda}/2}}{2K_{\bar{\lambda}}(\sqrt{\bar{p}\bar{\delta}})}z^{\bar{\lambda}-1}\exp\left(-\frac{1}{2}\left(\bar{\delta}z^{-1}+\bar{p}z\right)\right), \tag{68}$$

where $\bar{p} = \sqrt{\bar{\delta}^2(\bar{\alpha}^2 - \boldsymbol{\beta}'\boldsymbol{\Sigma}_X\boldsymbol{\beta})}$ and $\bar{\delta} = |\boldsymbol{\Sigma}_X|^{1/n}$. Define $W = X|Z$, then W is multivariate normal distributed,

$$W \sim N\left(\boldsymbol{\mu}_X + z\Delta_X\boldsymbol{\beta}, z\Delta_X\right), \tag{69}$$

and the unconditional distribution of X is MGH as (58) by Barndorff-Nielsen (1977).

Before calculating the MGF and the moments, we introduce a change of variable for the convenience of future calculations. Let V be a random vector distributed as the multivariate standard normal (MVSN) distribution. By the properties of the MGH distribution, X will have the same distribution law of,

$$X \overset{d}{=} z^{1/2}\left(z^{-1/2}\boldsymbol{\mu}_X + z^{1/2}\Delta_X\boldsymbol{\beta} + \mathbf{P}V\right),$$

where $\mathbf{P}\mathbf{P}' = \Delta_X$. Noting that $\boldsymbol{\mu}_X = \mathbf{0}$, and defining $Y = z^{1/2}\Delta_X\boldsymbol{\beta} + \mathbf{P}V$, the variable X can be denoted as,

$$X \overset{d}{=} z^{1/2}Y, \tag{70}$$

where Y is MGH distributed. Considering (70) the elliptical truncated region $C(X,a)$ can be transformed as,

$$\begin{aligned}C(X,a) = a \leq (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A) &\overset{d}{=} a \leq z^{1/2}(Y - z^{-1/2}\boldsymbol{\mu}_A)'\mathbf{A}(Y - z^{-1/2}\boldsymbol{\mu}_A)z^{1/2}\\ &\overset{d}{=} z^{-1}a \leq (Y - z^{-1/2}\boldsymbol{\mu}_A)'\mathbf{A}(Y - z^{-1/2}\boldsymbol{\mu}_A),\\ &\overset{d}{=} z^{-1}a \leq (Y - \boldsymbol{\mu}_{A;Y})'\mathbf{A}(Y - \boldsymbol{\mu}_{A;Y}),\\ &\overset{d}{=} C(z^{1/2}Y, a),\end{aligned} \tag{71}$$

where $\boldsymbol{\mu}_{A;Y} = z^{-1/2}\boldsymbol{\mu}_A$. Considering (70) and (71), define $W_Y = (W_{1;Y}, \ldots, W_{n;Y}) \equiv z^{1/2}Y|Z \equiv X|Z$, then W_Y is a random vector with MVN distribution,

$$z^{1/2}Y|Z \equiv W_Y \sim N\left(z^{1/2}\Delta_X\boldsymbol{\beta}, \Delta_X\right), \tag{72}$$

and

$$C(z^{1/2}Y, a)|Z \equiv C(W_Y, a). \tag{73}$$

The truncated MGF of X at the ellipsoid $C(X, a)$ applying (72) and (73) can be approximated as,

$$\mathbb{E}\left[\exp(\mathbf{t}X)|C(X,a)\right] \quad = \quad m(\mathbf{t}, C(X,a)) = \mathbb{E}_z[\mathbb{E}\left[\exp(\mathbf{t}W_Y)|C(W_Y,a)\right]]. \tag{74}$$

Using the results of Section 1 the internal expression of (74),

$$\mathbb{E}[\exp(\mathbf{t}W_Y)|C(W_Y,a)], \tag{75}$$

is the MGF of the MVN distribution, and can be calculated as,

$$\mathbb{E}[\exp(\mathbf{t}W_Y)|C(W_Y,a)] \quad = \quad L_z^{-1}\exp\left(\mathbf{t}'\left(z^{1/2}\Delta_X\boldsymbol{\beta}\right) + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}_X\mathbf{t}\right)H_{n;\mathbf{E},\mathbf{b}_z(\mathbf{t})}(z^{-1}a/p). \tag{76}$$

Then, the MGF expression (74) can be approximated by,

$$m(\mathbf{t}, C(X,a)) = \mathbb{E}_z\left[L_z^{-1}\exp\left(\mathbf{t}'\left(z^{1/2}\Delta_X\boldsymbol{\beta}\right) + \frac{1}{2}\mathbf{t}'\boldsymbol{\Sigma}_X\mathbf{t}\right)H_{n;\mathbf{E},\mathbf{b}_z(t)}(z^{-1}a/p)\right]. \tag{77}$$

If we set $\mathbf{t} = \mathbf{0}$ in (77), we have as a result that,

$$1 = \mathbb{E}_z\left[L_z^{-1}H_{n;\mathbf{E},\mathbf{b}_{0;z}}(z^{-1}a/p)\right], \tag{78}$$

and equality in (62) will hold if $\mathbb{E}_z[\cdot]$ exists. The truncated zeroth-order moment (probability) of X at the ellipsoid $C(X,a)$ is,

$$L = \mathbb{E}_z[L_z]. \tag{79}$$

To calculate (79), we need to develop the series (62),

$$\begin{aligned}
&\mathbb{E}_z\left[\sum_{i=0}^{\infty} G_{n+2i}(z^{-1}a/p)c_{i;0;z}\right] = \\
&\quad= \mathbb{E}_z\left[G_n(z^{-1}a/p)c_{0;0;z}\right] + \mathbb{E}_z\left[G_{n+2}(z^{-1}a/p)c_{1;0;z}\right] + \mathbb{E}_z\left[G_{n+4}(z^{-1}a/p)c_{2;0;z}\right] + \ldots, \\
&\quad= \mathbb{E}_z\left[G_n(z^{-1}a/p)\exp\left(-\frac{1}{2}\mathbf{b}'_{0;z}\mathbf{b}_{0;z}\right)\prod_{j=1}^{n}(p/e_j)^{1/2}\right] + \\
&\qquad \mathbb{E}_z\left[G_{n+2}(z^{-1}a/p)2^{-1}\exp\left(-\frac{1}{2}\mathbf{b}'_{0;z}\mathbf{b}_{0;z}\right)\prod_{j=1}^{n}(p/e_j)^{1/2}d_{1;0;z}\right] + \\
&\qquad \mathbb{E}_z\left[G_{n+4}(z^{-1}a/p)4^{-1}\exp\left(-\frac{1}{2}\mathbf{b}'_{0;z}\mathbf{b}_{0;z}\right)\prod_{j=1}^{n}(p/e_j)^{1/2}\left(d_{2;0;z} + 2^{-1}d_{1;0;z}^2\right)\right] + \ldots.
\end{aligned} \tag{80}$$

By definition we can apply the following factors over the terms dependent on z in (80),

$$\begin{aligned}
\mathbf{b}'_{0;z}\mathbf{b}_{0;z} &= \left(\mathbf{K}^{-1}\mathbf{P}^{-1}(z^{-1/2}\boldsymbol{\mu}_A - z^{1/2}\Delta_X\boldsymbol{\beta})\right)'\left(\mathbf{K}^{-1}\mathbf{P}^{-1}(z^{-1/2}\boldsymbol{\mu}_A - z^{1/2}\Delta_X\boldsymbol{\beta})\right) \\
&= z^{-1}\left(\boldsymbol{\mu}'_A\Delta_X^{-1}\mathbf{K}^{-2}\boldsymbol{\mu}_A\right) - 2\left(\Delta_X\boldsymbol{\beta}\right)'\left(\Delta_X^{-1}\mathbf{K}^{-2}\right)\boldsymbol{\mu}_A + z\left(\Delta_X\boldsymbol{\beta}\right)'\left(\Delta_X^{-1}\mathbf{K}^{-2}\right)\left(\Delta_X\boldsymbol{\beta}\right), \\
&= z^{-1}\left(\boldsymbol{\mu}'_A\Delta_X^{-1}\boldsymbol{\mu}_A\right) - 2\left(\Delta_X\boldsymbol{\beta}\right)'\left(\Delta_X^{-1}\right)\boldsymbol{\mu}_A + z\left(\Delta_X\boldsymbol{\beta}\right)'\left(\Delta_X^{-1}\right)\left(\Delta_X\boldsymbol{\beta}\right), \\
&= z^{-1}B_{0;-1} + B_{0;0} + zB_{0;1},
\end{aligned}$$

and,

$$d_{i;0;z} \quad = \quad dA_i + z^{-1} dB_i + z dC_i, \tag{81}$$

where,

$$\begin{aligned} B_{0;-1} &= \boldsymbol{\mu}_A' \Delta_X^{-1} \boldsymbol{\mu}_A, \\ B_{0;0} &= -2\boldsymbol{\beta}' \boldsymbol{\mu}_A, \\ B_{0;1} &= \boldsymbol{\beta}' \Delta_X \boldsymbol{\beta}, \end{aligned}$$

with,

$$\begin{aligned} dA_i &= \sum_{j=1}^{n} (1 - p/e_i)^i + ipB_{0;0} \sum_{j=1}^{n} (1/e_i)(1 - p/e_j)^{i-1}, \\ dB_i &= ipB_{0;-1} \sum_{j=1}^{n} (1/e_i)(1 - p/e_j)^{i-1}, \\ dC_i &= ipB_{0;1} \sum_{j=1}^{n} (1/e_i)(1 - p/e_j)^{i-1}, \end{aligned}$$

and in consequence developing the recurrence $c_{i;0;z}$ as in (6), (7), and (8) with the corresponding change (63), the coefficients $c_{i;0;z}$ can be denoted by a polynomial over two terms, $\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1} + B_{0;0} + zB_{0;1}\right)\right)$ and z by,

$$\begin{aligned} c_{i;0;z} = \exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1} + B_{0;0} + zB_{0;1}\right)\right) \times \\ \left(c_{i_{a_{-i}};0} z^{-i} + c_{i_{a_{-(i-1)}};0} z^{-(i-1)} + \cdots + c_{i_{a_0};0} z^0 + \cdots + c_{i_{a_{i-1}};0} z^{i-1} + c_{i_{a_i};0} z^i\right), \end{aligned} \tag{82}$$

with $c_{i_{a_j};0}, i, j \geq 0$ that are coefficients not dependent on z. The value of the coefficients $c_{i_{a_j};0}$ is found by equating (7) with (82) and substituting $\mathbf{b}(t)$ by $\mathbf{b}_{0;z}$, considering the relation derived in (81). Hence, the terms in the series (80) can be denoted by,

$$\begin{aligned} \mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)c_{i;0;z}\right] &= \mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)z^{-i}\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1} + B_{0;0} + zB_{0;1}\right)\right)\right] c_{i_{a_{-i}};0} + \cdots + \\ &\quad \mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)z^{0}\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1} + B_{0;0} + zB_{0;1}\right)\right)\right] c_{i_{a_0};0} + \cdots + \\ &\quad \mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)z^{i}\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1} + B_{0;0} + zB_{0;1}\right)\right)\right] c_{i_{a_i};0}. \end{aligned} \tag{83}$$

We calculate $\mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)z^j\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1} + B_{0;0} + zB_{0;1}\right)\right)\right]$ for $j \in \{-i, \ldots, 0, \ldots, i\}$ applying the definitions of chi-squared distribution and $\mathbb{E}_z$,

$$\begin{aligned} \mathbb{E}_z&\left[G_{n+2i}(z^{-1}a/p)z^j\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1} + B_{0;0} + zB_{0;1}\right)\right)\right] = \\ &= \int_0^\infty \int_{\eta a/p}^\infty \frac{x^{(n+2i)/2-1}\exp\left(-\frac{1}{2}x\right)}{2^{(n+2i)/2}\Gamma\left(\frac{n+2i}{2}\right)} \frac{(\psi/\chi)^{\lambda/2}\exp\left(-\frac{1}{2}(\chi z^{-1} + \psi z)\right)}{2K_\lambda(\sqrt{\chi\psi})} z^{j+\lambda-1} \times \\ &\quad \exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1} + B_{0;0} + zB_{0;1}\right)\right) dx dz, \end{aligned} \tag{84}$$

where $\chi = \bar{\delta}$, $\psi = \bar{p}$, and $\lambda = \bar{\lambda}$ is a different parametrisation of GIG variables commonly used in the literature. Applying the change of variable $z^{-1}y = x$ to (84) we have,

$$= \int_0^\infty \int_{a/p}^\infty \frac{(z^{-1}y)^{(n+2i)/2-1} \exp\left(-\frac{1}{2}z^{-1}y\right)}{2^{(n+2i)/2}\Gamma\left(\frac{n+2i}{2}\right)} \frac{(\psi/\chi)^{\lambda/2} \exp\left(-\frac{1}{2}(\chi z^{-1} + \psi z)\right)}{2K_\lambda(\sqrt{\chi\psi})} z^{j+\lambda-1} \times$$
$$\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1} + B_{0;0} + zB_{0;1}\right)\right) dydz$$
$$= \frac{(\psi/\chi)^{\lambda/2} \exp\left(-\frac{1}{2}B_{0;0}\right)}{2^{(n+2i)/2+1}\Gamma\left(\frac{n+2i}{2}\right) K_\lambda(\sqrt{\chi\psi})} \int_0^\infty \left(\int_{a/p}^\infty y^{(n+2i)/2-1} \exp\left(-\frac{1}{2}yz^{-1}\right) dy\right) z^{j+\lambda-(n+2i)/2-1} \times$$
$$\exp\left(-\frac{1}{2}\left((\chi + B_{0;-1})z^{-1} + (\psi + B_{0;1})z\right)\right) dz.$$

Introduce the change of variable $t = \frac{1}{2}yz^{-1}$,

$$= \frac{(\psi/\chi)^{\lambda/2} \exp\left(-\frac{1}{2}B_{0;0}\right)}{2^{(n+2i)/2+1}\Gamma\left(\frac{n+2i}{2}\right) K_\lambda(\sqrt{\chi\psi})} \int_0^\infty \left(\int_{\frac{1}{2}(a/p)z^{-1}}^\infty (2z)^{(n+2i)/2-1} t^{(n+2i)/2-1} \exp\left(-t\right) 2zdt\right) \times$$
$$z^{j+\lambda-(n+2i)/2-1} \exp\left(-\frac{1}{2}\left((\chi + B_{0;-1})z^{-1} + (\psi + B_{0;1})z\right)\right) dz, \quad (85)$$

and the internal integral in (85) over t is an upper-incomplete gamma function[6] that is denoted using its properties as,

$$= \frac{(\psi/\chi)^{\lambda/2} \exp\left(-\frac{1}{2}B_{0;0}\right)}{2^{(n+2i)/2+1}\Gamma\left(\frac{n+2i}{2}\right) K_\lambda(\sqrt{\chi\psi})} \int_0^\infty \left(2^{(n+2i)/2} z^{(n+2i)/2}\right) \Gamma\left(\frac{n+2i}{2}, \frac{1}{2}(a/p)z^{-1}\right) \times$$
$$z^{j+\lambda-(n+2i)/2-1} \exp\left(-\frac{1}{2}\left((\chi + B_{0;-1})z^{-1} + (\psi + B_{0;1})z\right)\right) dz$$
$$= \frac{(\psi/\chi)^{\lambda/2} \exp\left(-\frac{1}{2}B_{0;0}\right)}{2\Gamma\left(\frac{n+2i}{2}\right) K_\lambda(\sqrt{\chi\psi})} \int_0^\infty \left(\Gamma\left(\frac{n+2i}{2}\right) - \sum_{k=0}^\infty \frac{\left(\frac{1}{2}(a/p)z^{-1}\right)^{(n+2i)/2+k} \exp\left(-\frac{1}{2}(a/p)z^{-1}\right)}{\left(\frac{n+2i}{2}\right)\left(\frac{n+2i}{2}+1\right)\ldots\left(\frac{n+2i}{2}+k\right)}\right) \times$$
$$z^{j+\lambda-1} \exp\left(-\frac{1}{2}\left((\chi + B_{0;-1})z^{-1} + (\psi + B_{0;1})z\right)\right) dz. \quad (86)$$

Let $\chi_{Bap} = \chi + B_{0;-1} + (a/p)$, $\chi_B = \chi + B_{0;-1} + (a/p)$, $\psi_B = \psi + B_{0;1}$ and (86) is transformed in,

$$= \frac{(\psi/\chi)^{\lambda/2} \exp\left(-\frac{1}{2}B_{0;0}\right)}{2\Gamma\left(\frac{n+2i}{2}\right) K_\lambda(\sqrt{\chi\psi})} \left(\Gamma\left(\frac{n+2i}{2}\right) \int_0^\infty z^{j+\lambda-1} \exp\left(-\frac{1}{2}\left(\chi_B z^{-1} + \psi_B z\right)\right) dz - \right.$$
$$\left. \sum_{k=0}^\infty \frac{\left(\frac{1}{2}(a/p)\right)^{(n+2i)/2+k}}{\prod_{s=0}^k \left(\frac{n+2i}{2}+s\right)} \int_0^\infty z^{j+\lambda-(n+2i)/2-k-1} \exp\left(-\frac{1}{2}\left(\chi_{Bap} z^{-1} + \psi_B z\right)\right) dz\right). \quad (87)$$

[6]The definition of the upper-incomplete gamma function is,

$$\Gamma(x, y) = \int_y^\infty t^{x-1} \exp(-t),$$

and it can be denoted as,

$$\begin{aligned} \Gamma(x, y) &= \Gamma(x) - \gamma(x, y) \\ &= \Gamma(x) - \sum_{k=0}^\infty \frac{y^{x+k} \exp(-y)}{x(x+1)\ldots(x+k)}. \end{aligned}$$

The integrals over z in (87) are r-th moments of a GIG distributed random variable,

$$= \frac{(\psi/\chi)^{\lambda/2}\exp\left(-\frac{1}{2}B_{0;0}\right)}{2\Gamma\left(\frac{n+2i}{2}\right)K_\lambda(\sqrt{\chi\psi})}\left(\Gamma\left(\frac{n+2i}{2}\right)\left(\frac{\chi_B}{\psi_B}\right)^{(\lambda+j)/2}K_{\lambda+j}\left(\sqrt{\chi_B\psi_B}\right)-\right.$$
$$\left.\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}(a/p)\right)^{(n+2i)/2+k}}{\prod_{s=0}^{k}\left(\frac{n+2i}{2}+s\right)}\left(\frac{\chi_{Bap}}{\psi_B}\right)^{(\lambda+j-(n+2)/2-k)/2}K_{\lambda+j-(n+2i)/2-k}\left(\sqrt{\chi_{Bap}\psi_B}\right)\right). \quad (88)$$

Considering (80), (82), (83), and (88) the result (78) is derived.

We calculate the elliptical truncated first-order moment using the change of variable in (70) and (73),

$$\mathbb{E}\left[X|a \le (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] \quad = \quad \mathbb{E}_z[z^{1/2}\mathbb{E}[Y|z, z^{-1}a \le (Y-\boldsymbol{\mu}_{A;Y})'\mathbf{A}(Y-\boldsymbol{\mu}_{A;Y})]].$$

By Proposition (1.1), the internal expression is the elliptical truncated first-order moment of the MVN, and applying the definition of the cdf and the first-order moment of a GIG distribution we have,

$$\mathbb{E}\left[X|a \le (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] \quad = \quad \mathbb{E}_z\left[z^{1/2}L_z^{-1}\sum_{i=0}^{\infty}G_{n+2i}(z^{-1}a/p)c_{i;[\partial\mathbf{t};\mathbf{0}];z}\right]. \quad (89)$$

where coefficients $c_{i;[\partial\mathbf{t};\mathbf{0}];z}$ are equal to $c_{i;[\partial\mathbf{t};\mathbf{0}]}$ as in (22) substituting $\mathbf{b}_0$ by $\mathbf{b}_{0;z}$. The expected value in (89), is solved introducing $z^{1/2}$ inside the coefficients $c_{i;[\partial\mathbf{t};\mathbf{0}];z}$ and applying the decomposition in (82), (83) for $c_{i;[\partial\mathbf{t};\mathbf{0}];z}$,

$$c_{i;[\partial\mathbf{t};\mathbf{0}];z}z^{1/2} = \exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1}+B_{0;0}+zB_{0;1}\right)\right)\left(c_{i_{a_{-i}};[\partial\mathbf{t};0]}z^{-i+(1/2)}+c_{i_{a_{-(i-1)}};[\partial\mathbf{t};0]}z^{-(i-3/2)}+\dots\right.$$
$$\left.+c_{i_{a_0};[\partial\mathbf{t};0]}z^{1/2}+\dots+c_{i_{a_{i-1}};[\partial\mathbf{t};0]}z^{i-1/2}+c_{i_{a_i};[\partial\mathbf{t};0]}z^{i+1/2}\right), \quad (90)$$

and,

$$\mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)z^{1/2}c_{i;[\partial\mathbf{t};0];z}\right] =$$
$$\mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)z^{-i+(1/2)}\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1}+B_{0;0}+zB_{0;1}\right)\right)\right]c_{i_{a_{-i}};[\partial\mathbf{t};0]}+\dots+$$
$$\mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)z^{1/2}\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1}+B_{0;0}+zB_{0;1}\right)\right)\right]c_{i_{a_0};[\partial\mathbf{t};0]}+\dots+$$
$$\mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)z^{i+1/2}\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1}+B_{0;0}+zB_{0;1}\right)\right)\right]c_{i_{a_i};[\partial\mathbf{t};0]}. \quad (91)$$

The solutions to the internal integrals in (91) are solved as in (88),

$$\mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)z^{j+1/2}\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1}+B_{0;0}+zB_{0;1}\right)\right)\right] =$$
$$= \frac{(\psi/\chi)^{\lambda/2}\exp\left(-\frac{1}{2}B_{0;0}\right)}{2\Gamma\left(\frac{n+2i}{2}\right)K_\lambda(\sqrt{\chi\psi})}\left(\Gamma\left(\frac{n+2i}{2}\right)\left(\frac{\chi_B}{\psi_B}\right)^{(\lambda+j+1/2)/2}K_{\lambda+j+1/2}\left(\sqrt{\chi_B\psi_B}\right)-\right.$$
$$\left.\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}(a/p)\right)^{(n+2i)/2+k}}{\prod_{s=0}^{k}\left(\frac{n+2i}{2}+s\right)}\left(\frac{\chi_{Bap}}{\psi_B}\right)^{(\lambda+j-(n+2)/2-k+1/2)/2}K_{\lambda+j-(n+2i)/2-k+1/2}\left(\sqrt{\chi_{Bap}\psi_B}\right)\right), \quad (92)$$

therefore, having (89), (90), (91), and (92), the result on truncated first-order moments is obtained.

Finally, the elliptically truncated second-order moment is calculated,

$$\mathbb{E}\left[XX'|a \le (X-\boldsymbol{\mu}_A)'\mathbf{A}(X-\boldsymbol{\mu}_A)\right] \quad = \quad \mathbb{E}_z[z\mathbb{E}[YY'|z, z^{-1}a \le (Y-\boldsymbol{\mu}_{A;Y})'\mathbf{A}(Y-\boldsymbol{\mu}_{A;Y})]].$$

Applying the results of Proposition (1.1) we have,

$$\mathbb{E}\left[XX'|a \leq (X-\boldsymbol{\mu})'\boldsymbol{\Sigma}_X^{-1}(X-\boldsymbol{\mu})\right] \quad = \quad \mathbb{E}_z\left[zL_z^{-1}\sum_{i=0}^{\infty} G_{n+2i}(z^{-1}a/p)c_{i;[\partial\mathbf{t}\partial\mathbf{t};0];z}\right], \tag{93}$$

where coefficients $c_{i;[\partial\mathbf{t}\partial\mathbf{t};0];z}$ are equal to $c_{i;[\partial\mathbf{t}\partial\mathbf{t};0]}$ as in (26) substituting $\mathbf{b}_0$ by $\mathbf{b}_{0;z}$. The expected value in (93) is solved introducing z inside the coefficients $c_{i;[\partial\mathbf{t}\partial\mathbf{t};0];z}$ and applying the decomposition in (82) and (83) for $c_{i;[\partial\mathbf{t}\partial\mathbf{t};0];\eta}$,

$$c_{i;[\partial\mathbf{t}\partial\mathbf{t};\mathbf{0}];z}z = \exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1}+B_{0;0}+zB_{0;1}\right)\right)\left(c_{i_{a_{-i}};[\partial\mathbf{t}\partial\mathbf{t};0]}z^{-i+1} + c_{i_{a_{-(i-1)}};[\partial\mathbf{t}\partial\mathbf{t};0]}z^{-(i-2)} + \dots \right.$$
$$\left. + c_{i_{a_0};[\partial\mathbf{t}\partial\mathbf{t};0]}z + \dots + c_{i_{a_{i-1}};[\partial\mathbf{t}\partial\mathbf{t};0]}z^{i} + c_{i_{a_i};[\partial\mathbf{t}\partial\mathbf{t};0]}z^{i+1}\right), \tag{94}$$

and,

$$\begin{aligned}
&\mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)zc_{i;[\partial\mathbf{t}\partial\mathbf{t};0];z}\right] = \\
&\qquad \mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)z^{-i+1}\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1}+B_{0;0}+zB_{0;1}\right)\right)\right]c_{i_{a_{-i}};[\partial\mathbf{t}\partial\mathbf{t};0]} + \dots + \\
&\qquad \mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)z^{1}\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1}+B_{0;0}+zB_{0;1}\right)\right)\right]c_{i_{a_0};[\partial\mathbf{t}\partial\mathbf{t};0]} + \dots + \\
&\qquad \mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)z^{i+1}\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1}+B_{0;0}+zB_{0;1}\right)\right)\right]c_{i_{a_i};[\partial\mathbf{t}\partial\mathbf{t};0]}.
\end{aligned} \tag{95}$$

The solutions to the internal integrals in (95) are solved as in (88),

$$\begin{aligned}
&\mathbb{E}_z\left[G_{n+2i}(z^{-1}a/p)z^{j+1}\exp\left(-\frac{1}{2}\left(z^{-1}B_{0;-1}+B_{0;0}+zB_{0;1}\right)\right)\right] = \\
&= \frac{(\psi/\chi)^{\lambda/2}\exp\left(-\frac{1}{2}B_{0;0}\right)}{2\Gamma\left(\frac{n+2i}{2}\right)K_\lambda(\sqrt{\chi\psi})}\left(\Gamma\left(\frac{n+2i}{2}\right)\left(\frac{\chi_B}{\psi_B}\right)^{(\lambda+j+1)/2}K_{\lambda+j+1}\left(\sqrt{\chi_B\psi_B}\right) - \right. \\
&\qquad \left.\sum_{k=0}^{\infty}\frac{\left(\frac{1}{2}(a/p)\right)^{(n+2i)/2+k}}{\prod_{s=0}^{k}\left(\frac{n+2i}{2}+s\right)}\left(\frac{\chi_{Bap}}{\psi_B}\right)^{(\lambda+j-(n+2)/2-k+1)/2}K_{\lambda+j-(n+2i)/2-k+1}\left(\sqrt{\chi_{Bap}\psi_B}\right)\right),
\end{aligned} \tag{96}$$

then having (93), (94), (95), and (96) yields the result on truncated second-order moments.

Example 3. *Let X have the MGH distribution as in (58) with $\bar{\alpha} = 0.8, \bar{\lambda} = 0.7, \boldsymbol{\beta} = (0.1, 0.5)'$, $\boldsymbol{\mu}_X = (0,0)'$ and $\boldsymbol{\Sigma}_X$ defined as in Example 1. Let $a, \mathbf{a}, \mathbf{A}, \boldsymbol{\mu}_A$, and $C(\mathbf{x}, a)$ be defined as in Example 1. Applying Proposition 2.2, set $N = 250$, the zeroth-, first-, and second-order moments of X truncated at $C(\mathbf{x}, a)$ are,*

$$m_0(C(\mathbf{x},a)) = 0.8392, \qquad m_1(C(\mathbf{x},a)) = \begin{pmatrix} 0.4280 \\ 2.0778 \end{pmatrix}, \qquad m_2(C(\mathbf{x},a)) = \begin{pmatrix} 5.1339 & 3.2155 \\ 3.2155 & 10.8080 \end{pmatrix}.$$

To compare the results, we generated a Monte Carlo simulation with the following results,

$$m_0(C(\mathbf{x},a)) = 0.8396, \qquad m_1(C(\mathbf{x},a)) = \begin{pmatrix} 0.42584 \\ 2.0692 \end{pmatrix}, \qquad m_2(C(\mathbf{x},a)) = \begin{pmatrix} 5.0747 & 3.1776 \\ 3.1776 & 10.651 \end{pmatrix},$$

with the Monte Carlo simulation of X truncated at $C(\mathbf{x}, a)$ having a standard deviation of $(2.2255, 2.5560)'$ and a standard error of $(0.0070, 0.0080)'$.

4. Numerical application: quadratic forms in finance

Our results are theoretical over the distribution of elliptical truncated moments; we extend the Tallis (1963) results on normal elliptical truncated distributions for the general case where the centre of the elliptical truncation domain is not the centre of the distribution; for this, we use the Ruben (1962) expression, and then we extend his (1962) results for the MST and MGH distributions; nevertheless, we are interested in exploring the numerical properties of the expressions derived in Propositions 1.1, 2.2, and 3.1.

In this section we apply the analytical truncated moments derived for measuring risk functions in finance. Three approaches were used to solve the non-normal modelling of risks: (i) application of the copulae theory, (ii) non-parametric, and (iii) the use of more general non-elliptical distributions such as Lévy, stable, Pareto, and Pearson distributions. The first approach was motivated by the results of Klar (2002), and in Cherubini et al. (2004) there is a complete exposition of their application in finance. Nevertheless, the copulae modelling approach produces in the great majority of cases multivariate distributions where a closed-form of the joint density is unknown. Salem and Mount (1974), Madan and Seneta (1990), Aït-Sahalia and Lo (2000), Scaillet (2004), and Chen and Wang (2008) were among the first papers to explore models with parametric non-normal distributions, following the second approach. Salem and Mount (1974) analysed the gamma distribution, while Madan and Seneta (1990) the variance-gamma distribution. More recently, Eberlein et al. (1998) proposed the hyperbolic distribution for risk modelling, and Carr et al. (2002) tested an empirical modelling of the assets' returns with the hyperbolic, and variance-gamma distributions, to conclude that no diffusion component was present in the risk-neutral process of option data, but only jump-diffusion components.

We adopt the approach of modelling the assets' returns with a more general family of parametric distributions, calculating an analytic expression for the expected shortfall of quadratic portfolios when the assets' returns behave in accordance with the MVN, the MST distribution, and the family of the MGH distributions. The selection of the MGH distribution comes as a result of its tractability – considering MGH has an expression for the multivariate joint density, while copulae mehtods do not provide the closed or analytic expressions for the resulting multivariate density – and for its versatility for modelling, given that Student's t, hyperbolic, Laplace, normal-inverse Gaussian (NIG), normal-inverse gamma (NIGamm), normal-inverse chi-squared (NICH), and variance-gamma (VG) distributions are obtained from the MGH distribution after some parametrisation. Our results are an extension of the results of Broda (2012), where the expected shortfall of quadratic portfolios is calculated when the assets' returns have a Student's t distribution. Broda (2012) turns out to be an extension of the results of Glasserman et al. (2002), where Broda (2012) calculated the *VaR* for assets' returns with a Student's t distribution.

4.1. Quadratic forms in finance: Definitions

Let $S = (S_1, \ldots, S_n)$ be a random vector from the probabilistic space $(\Omega, \mathcal{F}, P)$. Let $\Delta S = X$, with distribution F_X. The random vector X represents n risk factors. Denote $\Delta S = S(0) - S(t)$ as the changes in the risk factors. The loss from a linear portfolio could be expressed as,

$$L_l = a_0 + \mathbf{a}'\Delta S, \tag{97}$$

and the loss from a quadratic portfolio,

$$L_q = a_0 + \mathbf{a}'\Delta S + \Delta S'\mathbf{A}\Delta S = L_l + \Delta S'\mathbf{A}\Delta S, \tag{98}$$

where $\mathbf{A}$ is a symmetric matrix. Define the distribution of L_q as: $P(L_q \le x) = F_{L_q}(x)$. The definitions for L_l are equivalent. Assume that the distribution function $F_{L_q}(x)$ is continuous for the time being, results for non-continuous distribution functions can be developed in future extensions. Let us define the quantile of L_q as in Rockafellar and Uryasev (2002),

$$x^{(\alpha)} = \inf\{x \in \mathbb{D}, \text{ such that } P(L_q \le x) \ge \alpha\},$$

where $\mathbb{D}$ is the domain of L_q. The value-at-risk (*VaR*) of L_q, at the confidence level α is,

$$VaR_\alpha(L_q) = x^{(\alpha)}, \tag{99}$$

and the expected shortfall is defined as,

$$ES_\alpha(L_q) = \mathbb{E}[L_q | L_q \geq x^{(\alpha)}], \tag{100}$$

but (100) is the truncated first-order moment of the loss distribution L_q at the VaR_α threshold. This fact is well demonstrated in Acerbi and Tasche (2002), and Delbaen (2002); as for the definition of the ES_α they define concepts associated with truncated moments such as the tail conditional expectation (TCE), worst conditional expectation (WCE), and the tail mean (TM). In the case of continuous distributions the definition of the ES_α is equal to TCE, WCE, and TM. The distribution of L_q is univariate; however, L_q is a quadratic function of the multivariate random vector X, that has the multivariate distribution F_X, and for this reason we will be interested in exploring multivariate truncated moments.

Assume in this section that we have a random X that is multivariate normally distributed. Let us decompose (100) as,

$$\begin{aligned} ES_\alpha(L_q) &= a_0 + \mathbb{E}[\mathbf{a}'X | \mathbf{a}'X + X'\mathbf{A}X \geq x^{(\alpha)} - a_0] + \\ &\quad \mathbb{E}[X'\mathbf{A}X | \mathbf{a}'X + X'\mathbf{A}X \geq x^{(\alpha)} - a_0]. \end{aligned} \tag{101}$$

The term $\mathbf{a}'X + X'\mathbf{A}X$ of the condition could be rewritten as,

$$\mathbf{a}'X + X'\mathbf{A}X = (X - \boldsymbol{\mu}_A)' \mathbf{A} (X - \boldsymbol{\mu}_A) - \frac{1}{4}\mathbf{a}'\mathbf{A}^{-1}\mathbf{a}, \tag{102}$$

where $\boldsymbol{\mu}_A = -\frac{1}{2}\mathbf{a}'\mathbf{A}^{-1}$. The function in (102) is an ellipsoid, centred in $\boldsymbol{\mu}_A$ and translated by $b_0 = \frac{1}{4}\mathbf{a}'\mathbf{A}^{-1}\mathbf{a}$. Then, (101) could be expressed as,

$$\begin{aligned} ES_\alpha(L_q) &= a_0 + \mathbb{E}\left[\mathbf{a}'X \middle| (X - \boldsymbol{\mu}_A)' \mathbf{A} (X - \boldsymbol{\mu}_A) \geq x^{(\alpha)} - a_0 + b_0\right] + \\ &\quad \mathbb{E}\left[X'\mathbf{A}X \middle| (X - \boldsymbol{\mu}_A)' \mathbf{A} (X - \boldsymbol{\mu}_A) \geq x^{(\alpha)} - a_0 + b_0\right]. \end{aligned} \tag{103}$$

The expression in (103) is the sum of: (i) the constant a_0, (ii) the sum $\mathbf{a}'X$ of truncated first-order moments of X with an ellipsoid restriction, and (iii) the sum $X'\mathbf{A}X$ of truncated second-order moments of X with an ellipsoid restriction.

Let the vector of truncated first-order moments of X at the ellipsoid $C(\mathbf{x}, a)$ be denoted as,

$$\begin{aligned} m_1(C(\mathbf{x}, a)) &= (\mathbb{E}[X_1 | C(\mathbf{x}, a)], \ldots, \mathbb{E}[X_n | C(\mathbf{x}, a)]) \\ &= (m_{1;1}(C(\mathbf{x}, a)), \ldots, m_{1;n}(C(\mathbf{x}, a))), \end{aligned}$$

and the matrix of truncated second-order moments at $C(\mathbf{x}, a)$,

$$\begin{aligned} m_2(C(\mathbf{x}, a)) &= \begin{pmatrix} \mathbb{E}\left[X_1^2 | C(\mathbf{x}, a)\right]^2 & \cdots & \mathbb{E}[X_1 X_n | C(\mathbf{x}, a)] \\ \vdots & \ddots & \vdots \\ \mathbb{E}[X_n X_1 | C(\mathbf{x}, a)] & \cdots & \mathbb{E}\left[X_n^2 | C(\mathbf{x}, a)\right] \end{pmatrix}, \\ &= \begin{pmatrix} m_{2;1,1}(C(\mathbf{x}, a)) & \cdots & m_{2;1,n}(C(\mathbf{x}, a)) \\ \vdots & \ddots & \vdots \\ m_{2;n,1}(C(\mathbf{x}, a)) & \cdots & m_{2;n,n}(C(\mathbf{x}, a)) \end{pmatrix}, \end{aligned}$$

and by definition (103) the expected shortfall is equal to,

$$\begin{aligned} ES_\alpha(L_q) &= a_0 + \mathbb{E}\left[\sum_{i=1}^{n} a_i X_i \middle| (X - \boldsymbol{\mu})' \mathbf{A} (X - \boldsymbol{\mu}) \geq x^{(\alpha)} - a_0 + b_0\right] + \\ &\quad \mathbb{E}\left[\sum_{i=1}\sum_{j=1} a_{i,j} X_i X_j \middle| (X - \boldsymbol{\mu})' \mathbf{A} (X - \boldsymbol{\mu}) \geq x^{(\alpha)} - a_0 + b_0\right] \end{aligned}$$

Table 1: Description of Portfolios

This table displays the different options' portfolios used to test the analytic formulae of the multivariate truncated MGF, zeroth-, first-, and second-order moments.

Portfolio	Description
1	Short 2 puts and 2 calls (1 per each of 4 assets), 0.5y maturity, zero correlation.
2	Long 2 puts and 2 calls (1 per each of 4 assets), 0.5y maturity, zero correlation.
3	Same as 1, plus (δ_i) shares per asset (delta-hedged).
4	Same as 2, plus (δ_i) shares per asset (delta-hedged).
5–8	Same as 1–4, but with equicorrelated assets ($\rho = 0.5$).
9–16	Same as 1–8, but with a maturity of 1 month.

$$\begin{aligned}
&= a_0 + \mathbb{E}\left[a_1 X_1 \middle| (X-\boldsymbol{\mu})' \mathbf{A} (X-\boldsymbol{\mu}) \geq x^{(\alpha)} - a_0 + b_0\right] + \\
&\qquad \mathbb{E}\left[a_2 X_2 \middle| (X-\boldsymbol{\mu})' \mathbf{A} (X-\boldsymbol{\mu}) \geq x^{(\alpha)} - a_0 + b_0\right] + \dots \\
&\qquad \mathbb{E}\left[a_{1,1} X_1^2 \middle| (X-\boldsymbol{\mu})' \mathbf{A} (X-\boldsymbol{\mu}) \geq x^{(\alpha)} - a_0 + b_0\right] + \dots \\
&\qquad \mathbb{E}\left[a_{n,n} X_n^2 \middle| (X-\boldsymbol{\mu})' \mathbf{A} (X-\boldsymbol{\mu}) \geq x^{(\alpha)} - a_0 + b_0\right] \\
&= a_0 + a_1 m_{1;1}(C(\mathbf{x}, a)) + a_2 m_{1;2}(C(\mathbf{x}, a)) + \dots + a_{1,1} m_{2;1,1}(C(\mathbf{x}, a)) + \dots + a_{n,n} m_{2;n,n}(C(\mathbf{x}, a)) \\
&= a_0 + \mathbf{a}' m_1(C(\mathbf{x}, a)) + \mathbf{1}'(\mathbf{A} \odot m_2(C(\mathbf{x}, a)))\mathbf{1}. \qquad (104)
\end{aligned}$$

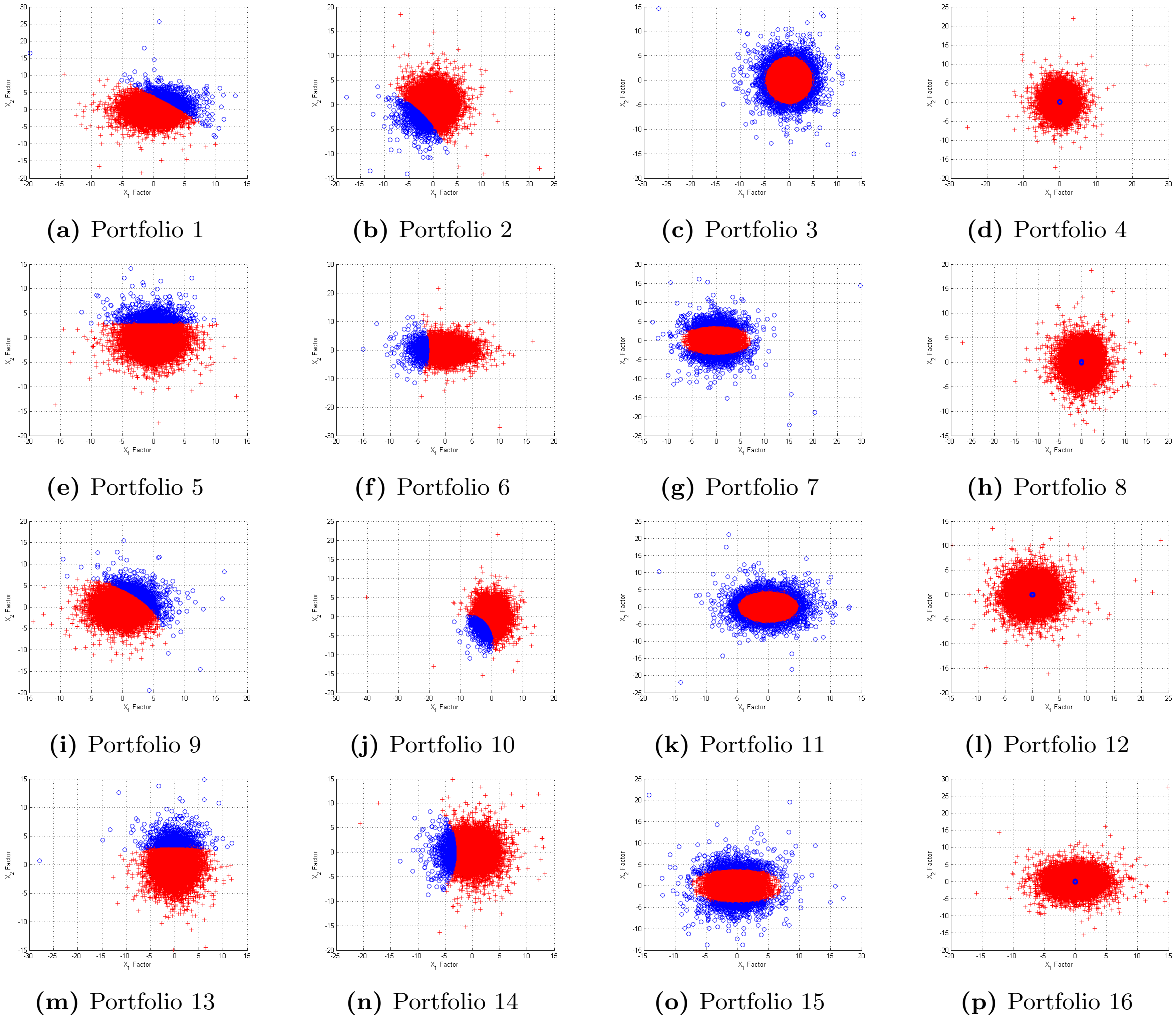
4.2. Numerical example: portfolio of financial derivatives

In this section we calculate the expected shortfall of a portfolio of options using (104). The portfolio is defined in Table 1, and it was first defined by Glasserman et al. (2002) when it was used to calculate the expected shortfall of a portfolio of normal distributed risk factors. (Broda, 2012) used this portfolio to calculate the expected shortfall of a portfolio of Student's t distributed risk factors. We tested three different distributions: MVN, MST, and MGH.[7] We calculated the VaR and the ES for a one-day risk horizon. The parameters used for the simulation of the sixteen (16) different portfolios were: initial stock price of 100, annual volatility of 30% for the stocks, risk-free interest rate of 5%, time to maturity of the options of 252 days (1 year) and 20 days (1 month), and uncorrelated ($\rho = 0$) and equi-correlated ($\rho = 0.5$) cases. For the MST distributed portfolios we used $\nu = 5$ degrees of freedom. For the MGH portfolios we used $\alpha = 2, \lambda = 0.05, \boldsymbol{\beta} = (0.02, \dots, 0.02)'$.

In Figure 1 we observe the bivariate region of ellipsoid truncation in the case of the MST distribution for each of the sixteen (16) different portfolios. Portfolios 1, 2, 5, 6, 9, 10, 13, and 14 correspond to cases where the truncation domain divides the distribution in two open regions. Portfolios 3, 7, 11, and 15 correspond to cases where the truncation domain is the outer tail of the distribution (open domain), and portfolios 4, 8, 12, and 16 correspond to cases where the truncation domain is a small inner region (closed domain).

Tables 2, 4, and 4 show the results of calculating the portfolios by the analytic methods of Propositions 1.1, 2.2, and 3.1 in column ANA (c), and we have a comparison of the results with: a Monte Carlo simulation

[7] This numerical application assumes for each of the cases that the underlying distribution corresponds to one of these three multivariate distributions, and uses the Black and Scholes (1973) model to price the portfolio; then in the case of the MST and MGH we are using an incorrect model for pricing; nevertheless, this is done with the intention of testing the analytic expression and comparing it with the Monte Carlo and the asymptotic expansion in similar way to Broda (2012).



(a) Portfolio 1 **(b)** Portfolio 2 **(c)** Portfolio 3 **(d)** Portfolio 4

(e) Portfolio 5 **(f)** Portfolio 6 **(g)** Portfolio 7 **(h)** Portfolio 8

(i) Portfolio 9 **(j)** Portfolio 10 **(k)** Portfolio 11 **(l)** Portfolio 12

(m) Portfolio 13 **(n)** Portfolio 14 **(o)** Portfolio 15 **(p)** Portfolio 16

Figure 1: Bivariate truncated domain of the 16 portfolios expected shortfall

The figures show the truncated domain that represents the area where the expected shortfall of the sixteen (16) portfolios for a bivariate Student's t distribution is calculated. The figures are generated by a scatter plot of a bivariate Monte Carlo simulation. In blue we have simulated points that fall in the truncated domain for which we have to calculated the moments, in red we have the simulated points that fall outside the truncated region.

(MC (a)) of the multivariate distribution of the risk factors, a Monte Carlo (MC2 (b)) simulation of the univariate distribution of the loss in the case of MVN distributions, and the asymptotic expansion of Broda (2012) (SPA2(b)) in the case of the MST and MGH distributions. The results of the expected shortfall using the analytic formula for MVN distributed risk factors (Proposition 1.1) shows an error below 1% and below 0.01% in some cases (portfolios 4, 8, 12, and 16). In the case of the expected shortfall using the analytic formula for MST distributed risk factors (Proposition 2.2), the error is below 5%, and it is quite similar to the asymptotic expansion of Broda (2012) (SPA2 (b)), and is even lower in most of the portfolios (2, 4, 5, 6, 7, 8, 10, 13 and 14). The results of the expected shortfall using the analytic formula for MGH distributed risk factors (Proposition 3.1), show a larger difference against the Monte Carlo calculated values. For some portfolios it is less than 1% (4, 8, 9, 10, 11, 12, 14, and 16), but for there are portfolios on which the error is as large as 112% (Portfolio 13). In the particular case of analytic multivariate truncated moments of MGH distributions, the analytic formula in (96) used for the calculation has an internal series. Although the outer series can rapidly converge, the internal series in (96) is of a lower convergence rate for some cases and dependent on the Bessel function $K_{\lambda+j-(n+2i)/2-k+1}(\sqrt{\chi_{B_{ap}}\psi_B})$, that increases in complexity with the number of terms j of the outer series in (80).

We are interested in the convergence (running time vs. residual error) of the analytic series. Figure 2 shows the convergence rate for the sixteen (16) different cases in black. An average Monte Carlo convergence of the sixteen portfolios is shown in blue. We can observe that for some portfolios, the analytic expression rapidly converges, in some cases even in just one iteration (portfolios 4, 12, and 15); nevertheless, there are cases where the convergence is very slow (portfolios 1, 2, 5, 6, and 13). Analysing these cases, we find that the analytic elliptical truncated moments' formulae can have a slow convergence rate in truncation cases that divide the distribution into two open regions (open domains) and have a large domain proportional to the total distribution. The Monte Carlo simulation average convergence rate is in between the fastest and the slowest analytic formulae cases.

5. Extreme cases

In this section we test the analytic multivariate elliptical truncated MVN, MST, and MGH formulae for several dimensions ($n = 2, 3, 4$), generating random cases with extreme parameters. An extreme parameter is generated with truncation centres one standard deviation away from the centre of the distribution, and with a large open truncation domain ($> 80\%$). We generate fifty (50) random cases, and then we calculated the zeroth-, first-, and second-order elliptical truncated moment applying propositions 1.1, 2.2, and 3.1. The parameters for the: (i) MVN cases are $\boldsymbol{\mu}_X = (\mu_1, \ldots, \mu_n), \mu_i \sim U(-1, 1), \boldsymbol{\Sigma}_X = S + S'$, where $S = tril(\sigma_{i,j}), \sigma_{i,j} \sim N(0, 1)$ with $U(a, b)$ the uniform distribution between (a, b), and $tril(\cdot)$ the lower triangular matrix function; (ii) MST cases are $\boldsymbol{\mu}_X = (0, \ldots, 0)$, $\nu = round(U(5, 30))$, and $\boldsymbol{\Sigma}_X$ similar to the MVN case; (iii) MGH cases are $\alpha = 2+U(0,1)\times 0.1, \lambda = 0.01+U(0,1)\times 0.1, \boldsymbol{\beta} = (\beta_1, \ldots, \beta_n), \beta_i = U(0,1)\times 0.1+0.01, i \in \{1, \ldots, n\}$.

Tables 5, 6, and 7 show the results. We observe that although for low dimensions the convergent series of the analytic truncated moments for the MVN distribution are over 94% for the zertoh-, first-, and second-order moments, for larger dimensions the convergent series reduces to 74% for the second-order moment. In the case of MVN distributions, the analytic formula shows over 88% convergence, while it reduces to only 40% of the convergent series for second-order moments of larger dimensions. This reduction is due to the complete and incomplete $gamma(\cdot)$ functions in the internal series (37) of the analytic formula. In the case of the analytic elliptical truncated moments of the MGH distributions we find a convergence of over 74% for low dimensions, but as low as 14% for larger dimensions ($n = 4$). This low proportion of the convergent series is due to the internal series expansion (96) that requires the evaluation of modified Bessel functions of second order. Monte Carlo simulations is a superior method for larger dimensions ($n > 5$) in terms of numerical efficiency; nevertheless, having an analytic formula offers a superior theoretical result over which other formulae can be derived – for example, the Greeks of the portfolios.

In Figure 3 we show the convergence rate (running time vs. residual error) for the calculation of the zeroth-order moment (m_0). The running time of the numerical implementation of the analytic expansion shows that it is slower than the Monte Carlo simulation for the three cases (MVN, MST, and MGH); still,

Table 2: Approximation error of the options' portfolio analytic expected shortfall (MVN).
This table displays the approximation error in calculating the expected shortfall (ANA (c)) *of sixteen portfolios of options with an MVN distribution with the analytic multivariate truncated moments' method. We calculate the VaR of the loss distribution ($VaR_L^{(0.01)}$) with a Monte Carlo simulation, and then we use it as a bound to calculate the expected shortfall. We compare the results obtained with a Monte Carlo simulated expected shortfall* (MC (a))*, and a Monte Carlo simulation using the univariate distribution of the loss (*MC2 (b)*). The relative error between the three methods is reported in the last three columns.*

#	$VaR_L^{(0.01)}$	$ES_L^{(0.01)}$					
		MC (a)	MC2 (b)	ANA (c)	$(b-a)/a$	$(c-b)/b$	$(c-a)/a$
1	3.81	4.41	4.41	4.40	−0.00%	−0.23%	−0.23%
2	3.48	3.98	3.98	3.96	+0.00%	−0.42%	−0.42%
3	0.22	0.28	0.28	0.28	−0.13%	+0.29%	+0.16%
4	0.08	0.08	0.08	0.08	+0.00%	−0.01%	−0.01%
5	4.70	5.44	5.44	5.44	−0.00%	−0.08%	−0.08%
6	4.29	4.87	4.87	4.85	+0.00%	−0.44%	−0.44%
7	0.27	0.35	0.35	0.35	−0.12%	+1.11%	+0.98%
8	0.08	0.08	0.08	0.08	+0.00%	−0.01%	−0.01%
9	3.67	4.28	4.28	4.31	−0.00%	+0.52%	+0.51%
10	3.00	3.33	3.33	3.34	+0.00%	+0.15%	+0.15%
11	0.58	0.75	0.75	0.74	−0.12%	−0.80%	−0.92%
12	0.18	0.18	0.18	0.18	+0.00%	+0.00%	+0.00%
13	4.61	5.44	5.44	5.42	−0.00%	−0.23%	−0.23%
14	3.55	3.92	3.92	3.93	+0.00%	+0.14%	+0.14%
15	0.67	0.90	0.90	0.90	−0.12%	−0.34%	−0.46%
16	0.18	0.18	0.18	0.18	+0.00%	−0.00%	−0.00%

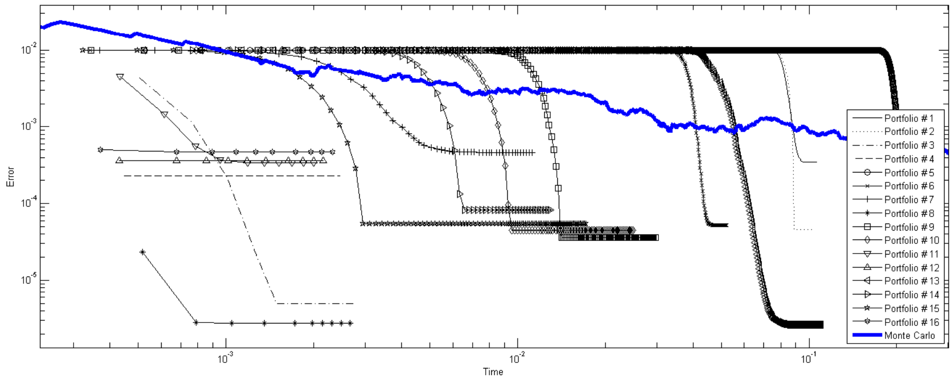
Table 3: Approximation error of the options' portfolio analytic expected shortfall (MST).

As Table 2, this table displays the approximation error calculating the expected shortfall (ANA (c)) *of sixteen portfolios of options with an MST distribution with the analytic multivariate truncated moments method. We calculate the VaR of the loss distribution ($VaR_L^{(0.01)}$) with a Monte Carlo simulation, and then we used as a bound for calculating the expected shortfall. We compare the obtained results with a Monte Carlo simulated expected shortfall* (MC (a)), *and an asymptotic expansion as in Broda (2012) (*SPA2 (b)*). The relative error between the three methods is reported in the last three columns.*

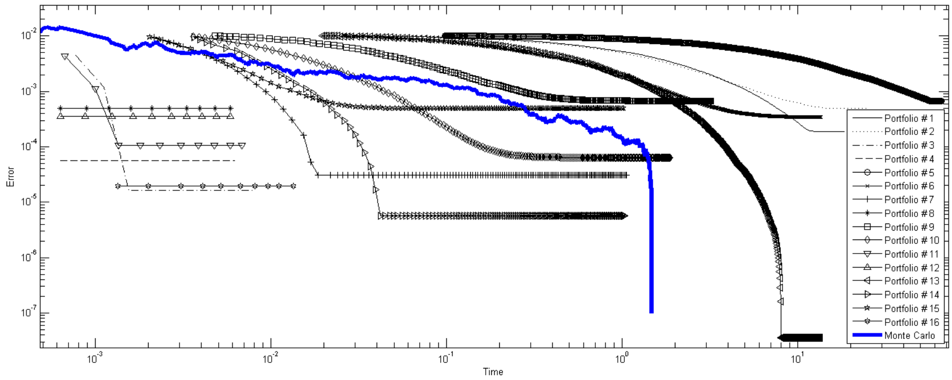
		$ES_L^{(0.01)}$					
#	$VaR_L^{(0.01)}$	MC (a)	SPA2 (b)	ANA (c)	$(b-a)/a$	$(c-b)/b$	$(c-a)/a$
1	4.32	5.69	5.71	5.94	+0.30%	+4.02%	+4.33%
2	3.93	4.88	4.81	5.02	−1.30%	+4.33%	+2.97%
3	0.42	0.79	0.85	0.83	+7.04%	−1.93%	+4.97%
4	0.08	0.08	0.08	0.08	−0.95%	+0.80%	−0.15%
5	5.35	7.17	7.02	7.37	−2.07%	+5.01%	+2.83%
6	4.77	6.05	5.85	6.07	−3.35%	+3.90%	+0.42%
7	0.46	0.90	0.89	0.90	−0.43%	+1.13%	+0.69%
8	0.08	0.09	0.08	0.08	−1.06%	+0.90%	−0.17%
9	4.45	6.01	6.07	6.38	+0.86%	+5.18%	+6.08%
10	3.21	3.89	3.77	3.85	−3.24%	+2.19%	−1.12%
11	1.08	1.96	2.14	2.09	+9.13%	−2.33%	+6.59%
12	0.18	0.19	0.18	0.18	−1.99%	+0.92%	−1.09%
13	5.33	7.81	7.43	7.65	−4.76%	+2.92%	−1.98%
14	3.82	4.48	4.46	4.51	−0.61%	+1.25%	+0.63%
15	1.23	2.18	2.26	2.38	+3.87%	+5.11%	+9.18%
16	0.18	0.19	0.18	0.18	−2.13%	+1.03%	−1.12%

Table 4: Approximation error of the options' portfolio analytic expected shortfall (MGH). *As Table 2 and , this table displays the approximation error calculating the expected shortfall* (ANA (c)) *of sixteen portfolios of options with an MGH distribution with the analytic multivariate truncated moments method. We calculate the VaR of the loss distribution ($VaR_L^{(0.01)}$) with a Monte Carlo simulation, and then we used as a bound for calculating the expected shortfall. We compare the obtained results with a Monte Carlo simulated expected shortfall* (MC (a))*, and a asymptotic expansion as in Broda (2012) (*SPA (b)*). The relative error between the three methods is reported in the last three columns.*

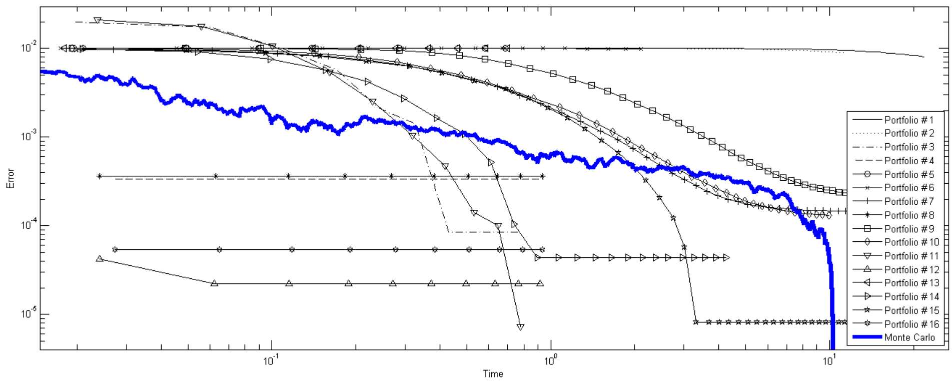
#	$VaR_L^{(0.01)}$	$ES_L^{(0.01)}$					
		MC (a)	SPA2 (b)	ANA (c)	$(b-a)/a$	$(c-b)/b$	$(c-a)/a$
1	5.17	6.48	6.46	4.56	−0.26%	−29.52%	−29.70%
2	4.41	5.31	5.30	6.89	−0.11%	+29.99%	+29.85%
3	0.54	0.78	0.79	0.79	+1.76%	−0.00%	+1.76%
4	0.08	0.08	0.08	0.08	+0.00%	+0.00%	+0.00%
5	6.59	8.30	8.12	1.20	−2.26%	−85.17%	−85.50%
6	5.32	6.39	6.30	7.37	−1.46%	+16.93%	+15.22%
7	0.62	0.94	0.90	0.82	−4.07%	−9.26%	−12.96%
8	0.08	0.08	0.08	0.08	+0.00%	+0.00%	+0.00%
9	5.27	6.80	6.76	6.74	−0.59%	−0.35%	−0.94%
10	3.55	4.07	4.10	4.10	+0.70%	+0.01%	+0.71%
11	1.35	1.96	1.97	1.97	+0.30%	−0.00%	+0.30%
12	0.18	0.18	0.18	0.18	+0.00%	−0.00%	+0.00%
13	6.78	8.78	8.58	-1.10	−2.29%	−112.77%	−112.47%
14	4.19	4.72	4.73	4.72	+0.15%	−0.16%	−0.02%
15	1.60	2.45	2.30	2.07	−6.09%	−10.09%	−15.57%
16	0.18	0.18	0.18	0.18	+0.01%	+0.00%	+0.01%



(a) Running time vs. residual error for the calculation of m_0 (MVN distribution).



(b) Running time vs. residual error for the calculation of m_0 (MST distribution).



(c) Running time vs. residual error for the calculation of m_0 (MGH distribution).

Figure 2: Running time vs. residual error of the options' portfolio truncated probability (m_0)

The figures show analytic and Monte Carlo simulation truncated moments' running time vs. residual error for the calculation of the truncated probability (m_0), for the sixteen (16) different portfolios' expected shortfall calculation. The results for the MVN, MST, and MGH distributions are plotted from the first to the third sub-figures. The Monte Carlo simulation convergence is the average of the sixteen (16) portfolios.

Table 5: Approximation error of analytic multivariate ellipsoid truncated moments – extreme cases (MVN distribution).

This table displays the approximation error calculating the Frobenius norm of the difference between the zeroth-, first-, and second-order moments of the Monte Carlo simulation and the analytic method for the different random variable dimension ($n = 2, 3, 4$) of fifty (50) sample extreme cases. The objective moments are calculated from samples generated from the multivariate ellipsoid truncated normal distribution. The standard errors of the mean values are reported in parentheses.

	Dimension					
	$n = 2$		$n = 3$		$n = 4$	
Analytic moment	Mean time	Mean error	Mean time	Mean error	Mean time	Mean error
$m_0(C_(\mathbf{x}, a))$	0.25s (0.0059)	0.0005 (0.0002) 96%	0.39s (0.0082)	0.0015 (0.0008) 92%	0.53s (0.0104)	0.0018 (0.0006) 96%
$m_1(C_(\mathbf{x}, a))$	1.52s (0.0313)	0.0016 (0.0002) 96%	3.43s (0.0782)	0.0032 (0.0010) 90%	6.16s (0.1263)	0.0061 (0.0013) 96%
$m_2(C_(\mathbf{x}, a))$	6.20s (0.1385)	0.0044 (0.0005) 94%	15.72s (0.3516)	0.0054 (0.0003) 86%	30.43s (0.5720)	0.0077 (0.0006) 74%
Total time	7.97s (0.1704)		19.53s (0.4192)		37.12s (0.6812)	
Monte Carlo time	6.03s (0.1183)		6.08s (0.1220)		6.24s (0.1441)	

Table 6: Approximation error of analytic multivariate ellipsoid truncated moments – extreme cases (MST distribution).

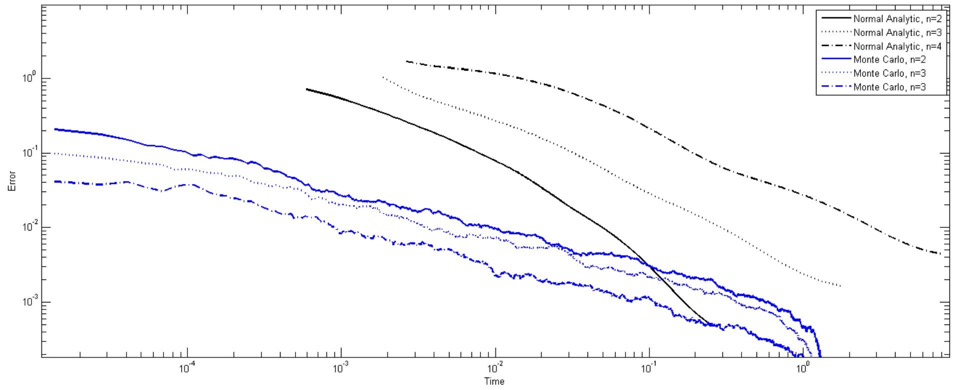
This table displays the approximation error calculating the Frobenius norm of the difference between the zeroth-, first-, and second-order moments of the Monte Carlo simulation and the analytic method for different random variable dimension ($n = 2, 3, 4$) of fifty (50) sample extreme cases. The objective moments are calculated from samples generated from multivariate ellipsoid truncated Student's t distribution. Standard errors of the mean values are reported in parentheses.

	Dimension					
	$n = 2$		$n = 3$		$n = 4$	
Analytic moment	Mean time	Mean error	Mean time	Mean error	Mean time	Mean error
$m_0(C_(\mathbf{x}, a))$	19.17s (0.2357)	0.0008 (0.0004) 92%	19.42s (0.2639)	0.0010 (0.0003) 72%	19.34s (0.2365)	0.0055 (0.0013) 66%
$m_1(C_(\mathbf{x}, a))$	0.01s (0.0005)	0.0026 (0.0006) 92%	0.30s (0.2921)	0.0040 (0.0009) 72%	0.02s (0.0004)	0.0083 (0.0014) 64%
$m_2(C_(\mathbf{x}, a))$	0.00s (0.0001)	0.0051 (0.0006) 88%	1.32s (1.3091)	0.0074 (0.0008) 58%	0.01s (0.0003)	0.0115 (0.0014) 40%
Total time	19.18s (0.2358)		21.04s (1.6096)		19.36s (0.2366)	
Monte Carlo time	7.18s (0.0911)		7.48s (0.1175)		7.49s (0.1135)	

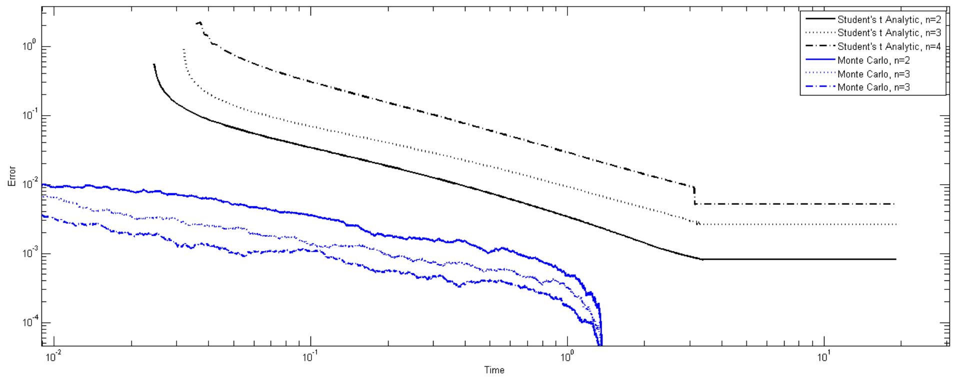
Table 7: Approximation error of analytic multivariate ellipsoid truncated moments – extreme cases (MGH distribution).

This table displays the approximation error calculating the Frobenius norm of the difference between the zeroth-, first-, and second-order moments of the Monte Carlo simulation and the analytic method for different random variable dimension ($n = 2, 3, 4$) of fifty (50) sample extreme cases. The objective moments are calculated from samples generated from multivariate ellipsoid truncated generalised hyperbolic distribution. Standard errors of the mean values are reported in parentheses.

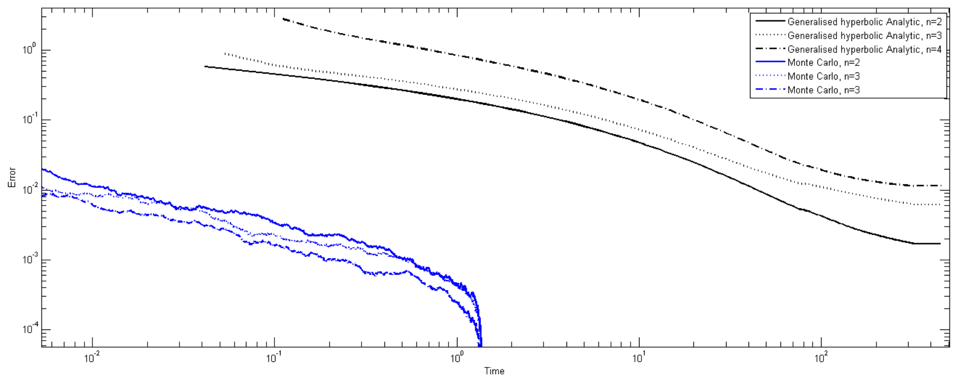
	Dimension					
	$n = 2$		$n = 3$		$n = 4$	
Analytic moment	Mean time	Mean error	Mean time	Mean error	Mean time	Mean error
$m_0(C_(\mathbf{x}, a))$	442.73s (6.3119)	0.0017 (0.0004) 76%	423.95s (5.6654)	0.0044 (0.0014) 54%	422.45s (5.6348)	0.0127 (0.0019) 36%
$m_1(C_(\mathbf{x}, a))$	1.89s (0.0329)	0.0062 (0.0010) 76%	2.08s (0.0421)	0.0072 (0.0007) 52%	2.38s (0.0425)	0.0128 (0.0012) 30%
$m_2(C_(\mathbf{x}, a))$	6.08s (0.1049)	0.0113 (0.0011) 74%	9.90s (0.2013)	0.0171 (0.0014) 46%	14.52s (0.2496)	0.0219 (0.0015) 14%
Total time	450.70s (6.3940)		435.94s (5.8346)		439.36s (5.8136)	
Monte Carlo time	11.93s (0.1717)		12.15s (0.2062)		12.03s (0.1969)	



(a) Running time vs. residual error for the calculation of m_0 – extreme cases (MVN distribution).



(b) Running time vs. residual error for the calculation of m_0 – extreme cases' (MST distribution).



(c) Running time vs. residual error for the calculation of m_0 – extreme cases (MGH distribution).

Figure 3: Running time vs. precision of the extreme cases truncated probability (m_0)

The figures shows Analytic and Monte Carlo simulation truncated moments average running time vs. average residual error for the calculation of the truncated probability (m_0) of fifty (50) sample extreme cases, for different random variable dimension ($n = 2, 3, 4$). Results for the MVN, MST, and MGH distributions are plot from the first, to the third sub-figures.

when we analyse the running time vs. the residual error in Figure 3, we observe that the decay rate of the residual error is similar between the Monte Carlo simulation and the analytic formula, but with a higher initial cost for the analytic formula. A numerical implementation in a faster programming language such as C++ is suggested as an extension of this study.

6. Conclusions

In this study we derived multivariate elliptical truncated moments of the MVN, MST, and MGH distributions. We derived an analytical formula for the MGF of the distributions, and then we derived the elliptical truncated MGF for the calculations. The analytic formulae extend the results of Tallis (1963) for radial truncation in the general case when the centre of the truncation region is different from the centre of the distributions. For the analytic derivations we used the Ruben (1962) results, then we extended these results for the MST and MGH distributions. The methodology used can be extended to a mixture of multivariate normal distributions. A numerical application for calculating quadratic forms in finance is presented, and numerical random extreme cases are tested. We find that the analytic formulae are convenient in numerical terms for low dimensions; however, the theoretical result of the expansion is still useful when an analytical formula is needed for further calculations such as the sensitivities of the elliptical truncated moments – the Greeks of quadratic portfolios. Further research is suggested to apply the same methodology for deriving formulae for other mixture of elliptical distributions, such as the skew-Normal or skew-Student's t distribution.

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AppendixA. Proof

Proposition AppendixA.1. *Let Z, X be as in Proposition 2.1, then,*

$$\mathbb{E}_\eta\left[\eta^{-i/2}\phi_n\left(\eta^{1/2}\mathbf{x};\mathbf{0},\boldsymbol{\Sigma}\right)\right]=\frac{\Gamma((\nu-i)/2)\nu^{\nu/2}}{2^{(i+n)/2}\Gamma(\nu/2)\Gamma(1/2)^n|\boldsymbol{\Sigma}|^{1/2}}\left(\mathbf{x}'\boldsymbol{\Sigma}_X^{-1}\mathbf{x}+\nu\right)^{-(\nu-i)/2}, \tag{A.1}$$

where $\mathbb{E}_\eta$ is the expected value conditional on the distribution of η.

Now we calculate the expectation on η using the definition,

$$\begin{aligned}\mathbb{E}_\eta\left[\eta^{-i/2}\phi_n\left(\eta^{1/2}\mathbf{x};\mathbf{0},\boldsymbol{\Sigma}\right)\right] &= \\ &= \int_{\eta=0}^{\infty}\eta^{-i/2}\left\{(2\pi)^{-n/2}|\boldsymbol{\Sigma}|^{-1/2}\exp\left(-\frac{\eta}{2}\mathbf{x}'\boldsymbol{\Sigma}_X^{-1}\mathbf{x}\right)\right\}\left\{\frac{1}{(2/\nu)^{\nu/2}}\frac{\eta^{\nu/2-1}\exp\left(-\frac{\nu}{2}\eta\right)}{\Gamma(\nu/2)}\right\}d\eta \\ &= \frac{\nu^{\nu/2}}{2^{\nu/2+n/2}\Gamma(1/2)^n\Gamma(\nu/2)|\boldsymbol{\Sigma}|^{1/2}}\int_{\eta=0}^{\infty}\eta^{-i/2+\nu/2-1}\exp\left(-\frac{\nu}{2}\eta\right)\exp\left(-\frac{\eta}{2}\mathbf{x}'\boldsymbol{\Sigma}_X^{-1}\mathbf{x}\right)d\eta,\end{aligned} \tag{A.2}$$

Then we apply the following change of variable $w=\frac{\eta}{2}\left(\mathbf{x}'\boldsymbol{\Sigma}_X^{-1}\mathbf{x}+\nu\right)$, and $d\eta=\frac{2}{\mathbf{x}'\boldsymbol{\Sigma}_X^{-1}\mathbf{x}+\nu}dw$ and (A.2) becomes,

$$\frac{\nu^{\nu/2}}{2^{\nu/2+n/2}\Gamma(1/2)^n\Gamma(\nu/2)|\boldsymbol{\Sigma}|^{1/2}}\int_{\eta=0}^{\infty}\left(\frac{2w}{\mathbf{x}'\boldsymbol{\Sigma}_X^{-1}\mathbf{x}+\nu}\right)^{-i/2+\nu/2-1}\exp(-w)\left(\frac{2}{\mathbf{x}'\boldsymbol{\Sigma}_X^{-1}\mathbf{x}+\nu}\right)dw. \tag{A.3}$$

Using the definition of the $\Gamma(\cdot)$ function in (A.3), the result follows.

Define the total probability,

$$L=\frac{\Gamma((\nu+n)/2)}{(\pi\nu)^{\nu/2}\Gamma(\nu/2)|\boldsymbol{\Sigma}|^{1/2}}\overset{(n)}{\int}_{C(\mathbf{x},a)}\left(1+\frac{1}{\nu}(\mathbf{x}-\boldsymbol{\mu})'\boldsymbol{\Sigma}_X^{-1}(\mathbf{x}-\boldsymbol{\mu})\right)^{-(\nu+n)/2},$$

where $C(\mathbf{x},a)$ is the ellipsoid defined in Proposition 1.1.