Discussion Paper

A Multi-asset Option Approximation for General Stochastic Processes

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A Multi-asset Option Approximation for General Stochastic Processes

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Abstract

We derived a model-free analytical approximation of the price of a multi-asset option defined over an arbitrary multivariate process, applying a semi-parametric expansion of the unknown risk-neutral density with the moments. The analytical expansion termed as the Multivariate Generalised Edgeworth Expansion (MGEE) is an infinite series over the derivatives of the known continuous time density. The expected value of the density expansion is calculated to approximate the option price. The expansion could be used to enhance a Monte Carlo pricing methodology incorporating the information about moments of the risk-neutral distribution. The numerical efficiency of the approximation is tested over a jump-diffusion density. For the known density, we tested the multivariate lognormal (MVLN), even though arbitrary densities could be used, and we provided its derivatives until the fourth-order. The MGEE relates two densities and isolates the effects of multivariate moments over the option prices. Results show that a calibrated approximation provides a good fit when the difference between the moments of the risk-neutral density and the auxiliary density are small relative to the density function of the former, and the uncalibrated approximation has immediate implications over risk management and hedging theory. The possibility to select the auxiliary density provides an advantage over classical Gram–Charlier A, B and C series approximations. The density approximation and the methodology can be applied to other fields of finance like asset pricing, econometrics, and areas of statistical nature.

1 Introduction

The distribution of the asset returns in equity markets is ‘fat-tailed’ and ‘skewed’ (Kraus and Litzenberger, 1976; Harvey and Siddique, 2000). For this reason, a semi-parametric formula like that of Jarrow and Rudd (1982) profoundly impacted the literature. They approximated an arbitrary continuous risk-neutral density of a univariate asset, using a Generalised Edgeworth Expansion (GEE) over a lognormal density. To obtain the option price, they integrated the resulting approximated density under the risk-neutral measure. To calibrate the approximation, the GEE requires the empirical moments of the unknown density of the asset. By doing this, not only the price is calculated, and the moments of the asset incorporated into the final formula, but also the effects of perturbations over the moments of the distribution on the option price can be easily observed.1

In this research, an approximate multi-asset option price is provided applying the Multivariate Generalised Edgeworth Expansion (MGEE) framework. In other words, we extend the results of Jarrow and Rudd (1982) to the multivariate case. Our formula disentangles the impact of multivariate higher-order moments on the option prices.2 It is the first time that a formula that disentangles the impact of multivariate higher-order moments on

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1As a result of the success of this model, it has been used in a large amount of empirical research, including Corrado and Su (1996), Bhandari and Das (2009) for options on portfolios, Lim et al. (2005) for a parametric option pricing model, Flamouris and Giamouridis (2002) for a semi-parametric model and Aït-Sahalia and Lo (1998) for a non-parametric model for density estimation.
2The option price formula is derived using a Fourier inversion method. Nevertheless, the method is applied for the large class of continuous density functions with partial derivatives, resulting in a formula that is on the time domain, and there will be no need of a Fourier inversion method for pricing. In a paper by Noguey and Perote (2008), a density expansion using the moments of the
option pricing has been provided. The main advantage of our approximation is that it is for arbitrary processes; this means it can be used with discontinuous-time models originating not only from a Wiener diffusion, but also from Levy processes like jump-diffusion. In the Jarrow and Rudd (1982) formula the value of the European option is equal to the Black and Scholes price plus corrections based on the difference of the moments of the lognormal distribution and the real market distribution. In this paper, the GEE is extended to the multivariate case (MGEE), and then the Black and Scholes price is calculated using a Monte Carlo simulation, as there exists no equivalent closed-form Black and Scholes formula for the multivariate case. An analysis of multi-asset model-free option pricing methodologies compared with structural methodologies could be developed with the MGEE. Another benefit of our results is that the moments of the risk-neutral density of the assets could be obtained separately through empirical work and, if they are available, then the price of the option is straightforwardly obtained using our formula. As a result, higher-order moment effects like the ones observed during market crashes can be easily modelled into the pricing or the hedging of the option.

The approximation provided allows us to calculate the moments of the distribution of the sum of lognormals in a multivariate setting, and this can be considered an interesting result not only for finance, but in general. In Ju (2002), an univariate approximation of the risk-neutral density is provided, using a Taylor expansion over a univariate lognormal density. Kristensen and Mele (2011) also provide an approximation with an application to asset pricing theory. This approximation is based on a Taylor expansion of a differential operator over the divergence between the Black and Scholes model price and the real price. Consequently, the moments of the distribution are not part of the option pricing formula, making it very difficult to understand how changes over the distribution affect the final price.

Our approach for valuing multi-asset options using the multivariate risk-neutral density is novel, since all previous models attempt to price multi-asset options with a function of univariate densities: Li et al. (2008) and Li et al. (2010) developed two new approximations, an original termed second-order boundary approximation, and an extension to the multivariate case of the Kirk (1996) formula for spread options termed the extended Kirk approximation. Both approximations reduce the dimensions of the problem, from a multivariate integration to a function of an univariate normal standard density. In Alexander and Venkatramanan (2011) the price of a spread option is approximated as the price of the sum of two compound options, and that is extended in Alexander and Venkatramanan (2012) for multi-asset options. The prices of the compound options were calculated by Geske (1979) and by Carr (1988). The final formula will be a function of the product of univariate densities. Working with the multivariate risk-neutral density requires additional notation from multivariate statistics. Nevertheless, the main advantage for empirical research is a more realistic framework, and it provides new tools for hedging and risk management.

The MGEE can be considered another important contribution of our research for other fields of application such as statistics. Although Perote (2004) and Del Brio et al. (2009) produced a Multivariate Edgeworth Expansion (MEE), this expansion is based on an approximation of the multivariate normal (MVN) distribution, with the complications of negative density values when the empirical density to fit is leptokurtic. We face the same risk, but if we select an appropriate distribution with skewness and kurtosis more similar to the risk-neutral density, this problem is diminished.

The structure of this paper is as follows: Section 2 examines the use of the MGEE approximation for hedging and risk management, and contains the definitions and the notation used. Section 3 describes the MGEE, and the method used in finding an approximation for multi-asset options. Section 4 presents the multi-asset option approximation. We integrate the resulting density from the MGEE using a Monte Carlo method. In Section 5, a numerical example is presented, where the MGEE is used to price an option over a multivariate jump-diffusion process and introduce the possible extensions in the use of the MGEE as a tool for risk management. In Section 6 we provide a calibration methodology. In Section 7 we present concluding remarks and further developments with some possible modifications to our approach.

distribution termed General Moments Expansion (GEM) is provided. This expansion generates only positive densities; however, it needs an additional vector of parameters of the same dimension of the distribution dimension; these additional parameters have no economical significance.

Schöggl (2013) provides an multi-asset option approximation using a multivariate Gram–Charlier A series expansion; however, there are assumptions over the risk-neutral density, and an additional methodology is needed to extract the moments inside the expansion from the Hermite polynomials.

Our results complement the results of Filipović et al. (2013), as we provide a thorough study of the higher-order moments effect over option prices. In Knight and Satchell (2001) a Gram–Charlier expansion is derived for pricing options using the first four moments of a univariate risk-neutral distribution.

Limpert et al. (2001) and Dufresne (2004) review the importance of the distribution of the sum of lognormals in finance, and in physical sciences in general.
2 Hedging the Risk-Neutral Density

A natural step for research on option pricing is the extension of all univariate models to the multivariate asset class. There exist popular versions of multi-asset options, one of which is the basket option: given a vector of weights \( \omega = \{ \omega_1, \ldots, \omega_n \} \), a strike price \( K \), and a \( n \)-variate vector of assets \( S(t) = \{ S_1, \ldots, S_n \} \), the payoff of a basket option is \( \Pi(S(t), \omega, K) = [\omega_1 S_1(t) + \cdots + \omega_n S_n(t) - K]^+ \). Rainbow, quanto, spread, Asian, and even index options can be regarded in the class of multivariate options. In general, the payoff of multi-asset options can be specified as a function of the assets: \( \Pi(S(t), H, K) = [H(S_1(t), \ldots, S_n(t), K)]^+ \), where \( H(\cdot) \) is a multivariate real function. A special case of basket options is the spread option, which is highly traded on NYMEX. The pricing formula provided and its derivation, the risk-free interest rate

The intuition is that an increase of the volatility of the risk-neutral density will increase the price of the option, and the increase in the skewness will reduce the price, and the increase of the kurtosis will again increase the price.

In our approximation, the changes of the risk-neutral density over the price can then be measured using this methodology. It is easier to estimate the moments of the Jarrow and Rudd (1982) approximation is that it links the difference of the cumulants of two distributions with the price of the option:

\[
C_0(\Pi(X)) = C_0(\Pi(S)) + 2nd \text{ order cumulants } \frac{\partial^2 g_S}{\partial S^2} \quad \text{3rd order cumulants } \frac{\partial^3 g_S}{\partial S^3} \quad \text{4th order cumulants } \frac{\partial^4 g_S}{\partial S^4}.
\]

The intuition is that an increase of the volatility of the risk-neutral density will increase the price of the option, an increase in the skewness will reduce the price, and the increase of the kurtosis will again increase the price.

In our approximation, the changes of the risk-neutral density over the price can then be measured using this methodology. It is easier to estimate the moments of \( f_X(t) \) by modelling risk-neutral asset prices as a multivariate variable using option market prices, than estimating the moments of the risk-neutral payoff function \( f_T \).

It is assumed that the asset risk-neutral density \( f_X(t) \) is unknown, but its moments are available. This density can be approximated using another known parametric density \( g_S(t) \), from which we can calculate the moments.

In this section we examine the use of multivariate risk-neutral densities for option pricing and hedging through some examples. We introduce some notation:

2.1 Model setup

In this section we define the arbitrary processes that can be approximated using a MGEE. Let \( X(t) = \{ X_i(t) \in \mathbb{R}^+, t \geq 0 \}, \ i \in \{ 1, \ldots, n \} \) be a general \( n \)-variate continuous stochastic process. This process is called the asset price process. Let \( Q \) be the \( n \)-variate risk-neutral probability measure. Denote by \( f_X(t) \) the existent and unique density of \( X(t) \) under \( Q \). We restrict \( X(t) \) to the class of processes where \( f_X(t) \) is a continuous density function, and its partial derivatives \( (df_X(t)/dX_i(t)) \) exist. Define the filtered probability space \( (\Omega, \mathcal{F}, Q) \), where \( \mathcal{F} \) is the filtration generated by the sigma-algebra \( \{ X_i(t), t \geq 0 \} \).

Additionally, define the \( n \)-variate stochastic process \( S = \{ S_i(t) \in \mathbb{R}^+, t \geq 0 \}, i \in \{ 1, \ldots, n \} \), described by:

\[
dS_i(t) = \mu_i S_i(t) dt + \sigma_i S_i(t) dW_i(t),
\]

\[
\langle dW_i(t), dW_j(t) \rangle = \rho_{i,j} dt,
\]

where \( i, j \in \{ 1, \ldots, n \} \), \( W_i(t) \) are Wiener processes under the risk-neutral measure \( Q \), and \( \mu_i, \sigma_i \) are the constant mean and the constant volatility of the variable \( S_i(t) \), and \( \rho_{i,j} \) is the constant correlation between \( S_i(t) \) and \( S_j(t) \). The process \( S(t) \) will be used to approximate the asset price process \( X(t) \), and has a multivariate lognormal density function \( g_S(t) \) under the risk-neutral measure \( Q \) (see Section 4.1). To simplify the option pricing formula provided and its derivation, the risk-free interest rate \( r \) will be considered constant.

The gist of the model approximation is to use the properties of the well-known distribution \( g_S(t) \) of the geometric Brownian motion (GBM) process \( S(t) \), to fit the unknown distribution \( f_X(t) \). In this sense we will have:

\[
f_X(t) = H(g_S(t)) + \varepsilon,
\]

where \( H \) is a function with information about the moments of \( f_X(t) \), and \( \varepsilon \) is a bounded error term. Denote by \( \Pi(X(t)) \) a payoff function over the asset price. Then, the price of the European option \( C_{t=0}(\Pi(X(t))) \) is the expected value of the discounted payoff under the risk-neutral measure:

\[
C_0(\Pi(X(t))) = \exp(-rt) \mathbb{E}_0^Q [\Pi(X(t))],
\]
2.2 Multi-asset options

The value of an option can be calculated with the expected value of the risk-neutral density. In the case of a basket option, where the payoff is given by,

\[ \Pi(X(t), \omega, K) = [\omega_1 X_1(t) + \cdots + \omega_n X_n(t) - K]^+ . \] (2)

It is important to mention that there are two possible approaches for calculating the expected value. The first approach is to compute the option price with the univariate density function \( f_X \) of the payoff \( \Pi(X(t), \omega, K) \); that is, a density function of the sum of the components \( \sum_{i=1}^{n} X_i(t) \). Then the expected value is found evaluating a single integral over the univariate risk-neutral density of the payoff:

\[ C_0(\Pi(X(t)), \omega, K) = \int_0^\infty \Pi(x(t)) dF_X, \] (3)

This is the dominant method used in the literature. An example is the basket option valuation of Margrabe (1978) and, more recently Ju (2002), Alexander and Venkatramanan (2012) and Li et al. (2010). In these articles, the density of the payoff function is usually modelled as a convolution of the sum of the densities of lognormal distributions.

The second approach is to integrate the payoff function over the assets’ multivariate risk-neutral density \( f_X \), and this implies the need to compute a multivariate integral:

\[ C_0(\Pi(X(t)), \omega, K) = \int_0^\infty \cdots \int_0^\infty \Pi(x(t)) dF_X(\omega, K). \] (4)

The univariate density of the sum is usually more complex to define than the multivariate risk-neutral density of the assets. For example, the probability density function (pdf) of the sum of lognormals is unknown, while there exists a known pdf for the multivariate lognormal density. However, the integral region of (4) will be a complex multivariate truncated region, while the integral region in (3) will be an easier univariate truncated one. We use the second method. Although it is more demanding and complex in the number of integrals and the region of integration, it will bring more information than using the first method.

2.3 Hedging and pricing with multivariate risk-neutral density

The fundamental reason for using the second approach is to provide additional information to the trader, hedger or risk manager about the price of the option. The univariate density of the sum of lognormals has turned out to be useful in the approximation of the option price, albeit it does not provide insights about the multivariate attributes of the risk-neutral density. Thus, for hedging it is imperative to use the multivariate density. Nevertheless, it seems counter-intuitive to use a univariate density for pricing, and a multivariate version for hedging. As an example, let us assume that a hedger uses a univariate sum of two lognormals to hedge an option:

Let \( \Pi(S(t), K) \) be the payoff function of a two-asset option, with price process \( C_0(\Pi(S(t), K)) \). Define the portfolio basket \( \Pi_p \):

\[ \Pi_p(S(t), K) = C_0(\Pi(S(t), K)) - \sum_{i=2}^{2} \Delta_i S_i(t), \] (5)

where the portfolio consists of one long position on \( C_0(\Pi(S(t), K)) \), the two-asset option price process at time \( t = 0 \), and short positions \( \Delta_i \), and the portfolio weights on each asset \( S_i(t) \). Applying Itô’s lemma, this portfolio will evolve by the process:

\[ d\Pi_p(S(t), K) = \left( \frac{\partial C_0(\Pi(S(t), K))}{\partial t} + \sum_{i=1}^{2} \sum_{j=1}^{2} \sigma_i \sigma_j S_i(t) S_j(t) \right) dt + \sum_{i=1}^{2} \left( \frac{\partial C_0(\Pi(S(t), K))}{\partial S_i(t)} \right) dS_i(t). \] (6)
Setting $\Delta_i = \frac{\partial C_0(\Pi(S(t), K))}{\partial S_i(t)}$, the portfolio in (5) will be risk-free:

$$\Pi_{P_1}(S(t), K) = C_0(\Pi(S(t), K)) - \sum_{i=2}^{n} \frac{\partial C_0(\Pi(S(t), K))}{\partial S_i(t)} S_i(t),$$

and (6) will lead to the multivariate Black and Scholes differential equation. Define a new asset, $X(t) = S_1(t) + S_2(t)$. The distribution of this asset will be the distribution of the sum of two lognormals. Setting a portfolio as in (5), the risk-free portfolio will be:

$$\Pi_{P_1}(X(t), K) = C_0(\Pi(X(t), K)) - \frac{\partial C_0(\Pi(X(t), K))}{\partial X(t)} X(t).$$

As hedging in real-world applications can be applied only in discrete spaces of time, after a discrete jump of time $\delta t, \delta > 0$, the difference of the portfolio processes (7) and (8) will be determined only by the difference in the positions for each asset:

$$\epsilon_{\Delta}(t, t + \delta t) = d\Pi_{P_1}(S(t + \delta t), K) - d\Pi_{P_1}(X(t + \delta t), K) = \sum_{i=1}^{2} \left( \frac{\partial C_0(\Pi(S(t), K))}{\partial S_i(t)} - \frac{\partial C_0(\Pi(X(t), K))}{\partial X(t)} \right) dS_i(t) = \epsilon_{\Delta}(t) dS_i(t).$$

The difference $\epsilon_{\Delta}(t, t + \delta t)$ is the tracking error of using the portfolio $\Pi_{P_1}(X(t), K)$ with the univariate density. This error will be zero only if we use the appropriate distribution of the sum of lognormals to hedge, the one that is still unknown in the literature at the time of the writing of this research.\footnote{Even when the multivariate density is used for hedging, in a two-volatilities world, if we swap the volatility of the asset $S_1(t)$ with the volatility of the asset $S_2(t)$, the price of the option will remain the same, although the hedge positions need to be changed, a counter-intuitive signal for a risk manager.}

For the general case with $n$-assets, define $\sigma_i = \sigma, i = \{1, \ldots, n\}$ as the volatility of the assets, and $\mathbb{E}[dS_i(t)] = (r - \frac{1}{2}\sigma^2) dt$ as the expected drift of the assets. Assume that the difference $\epsilon_{\Delta}(t) = \epsilon_i, i \in \{1, \ldots, n\}$ is constant for $t$. The expected value of the tracking error over the time $(0, t)$ will be:

$$\mathbb{E}[\epsilon_{\Delta}(0, t)] = \epsilon_c \left( r - \frac{1}{2}\sigma^2 \right) dt,$$

and the variance:

$$\mathbb{V}[\epsilon_{\Delta}(0, t)] = \epsilon_c^2 \left( n\sigma + \sum_{i=1}^{n} \sum_{i=1}^{n} \rho_{i,j} \sigma^2 \right) dt.$$
where $X$ of its components is defined as:

\[ C_0(\Pi(S(t), K)) = \exp(-rt)\mathbb{E}_Q^S [S(t) - K]^+ \]

\[ = \exp(-rt)\mathbb{P}^Q(S(t) \geq K) \left( \mathbb{E}_Q^S |S(t)| \mathbb{P}(S(t) \geq K) - K \right). \tag{9} \]

But, $\mathbb{E}_Q^S |S(t)| \mathbb{P}(S(t) \geq K)$ is the first moment of the variable $S(t)$ truncated at $K$, and $\mathbb{P}^Q(S(t) \geq K)$ is the zero-th moment. Then, to value multi-asset European options we can use the theory of multivariate truncated moments. The results of Rosenbaum (1961), Tallis (1961), Finney (1963), and Arismendi (2013) could be used for this purpose.

3 THE Multivariate Generalised Edgeworth Expansion (MGEE): THE DISTRIBUTION APPROXIMATION

Before defining the distribution approximation we introduce tensor notation with the purpose of simplifying the final formula. Attempting to extend Jarrow and Rudd’s (1982) results to the multivariate case without this notation would make the task intractable. We use the notation used by Kendall (1947) to provide general results. To simplify the notation, when the time index of an stochastic process is omitted we refer to the random variable at time $t$: $X \equiv X(t)$.\footnote{The vector notation $x(t)$ in lower-case refers to the variable of integration, then $E(X) \equiv \int x dF_x$.}

To define the tensor notation we use the summation convention as it is the appropriate notation for working with tensors. This notation is commonly used in physics and is attributed to Einstein. A tensor is a mathematical object similar to a multidimensional array. We use the brackets on the left-hand side to highlight the use of this implicit summation convention:

**Definition 3.1.** Let $a$ be a real valued vector of dimension $m$ with components $a_1, \ldots, a_m$. A tensor product of $X$ and $a$ between $p$ of their components is defined as:

\[ a_{[i_1]} \cdots a_{[i_p]}X_{[i_1]} \cdots X_{[i_p]} \equiv \sum_{i_1=1}^n \cdots \sum_{i_p=1}^n a_{i_1} \cdots a_{i_p}X_{i_1} \cdots X_{i_p}, \]

where $i_1, \ldots, i_p \in \{1, \ldots, n\}$ and the subscript $[i_p]$ represents a summation notation used to substitute the

Figure 1: Effects of hedging with univariate risk-neutral densities vs. multivariate risk-neutral densities.
Definition 3.2. Define the abbreviated integral operator as:

\[ k_i \] for \( F \) and that the cumulative distribution function

Definition 3.3. Let \( d \) distributions especially cannot be approximated with the MVN. This method has the inconvenience that only a limited set of distributions can be modelled, and heavy-tailed distributions can be included, but a more formal presentation using measure theory outside the scope of this paper will be required. There have been previous attempts to approximate a distribution using other distributions: the multivariate Gram--Charlier and the multivariate Edgeworth expansion with the Edgeworth--Sargan density will be required. There have been previous attempts to approximate a distribution using other distributions: the multivariate Gram--Charlier and the multivariate Edgeworth expansion with the Edgeworth--Sargan density of Perote (2004). However, in these cases the approximation is done over the multivariate normal distribution. This method has the inconvenience that only a limited set of distributions can be modelled, and heavy-tailed distributions especially cannot be approximated with the MVN.

Definition 3.3. Let \( X \) have an absolutely continuous density function \( f_X \). We assume that \( f_X \) is differentiable and that the cumulative distribution function \( F_X \) exists. Let \( I = \{i_1, \ldots, i_p\} \) be a vector of integer numbers, the \( p \)-order moment function of \( X \) is defined by,

\[
m_{i_1, \ldots, i_p}(x) = m_{p,I}(x) = \mathbb{E}[X_{i_1} \times \cdots \times X_{i_p}],
\]

and these moments can be computed with the integral:

\[
m_I(x) = \int_{-\infty}^{\infty} \frac{x_{i_1} \cdots x_{i_p} f_X}{F_X} dx_{i_1} \cdots dx_{i_p}.
\]

Another equivalent expression for moments is:

\[
m_{\alpha}(x) = E[X_1^{\alpha_1} X_2^{\alpha_2} \cdots X_n^{\alpha_n}],
\]

where \( \alpha \) is a vector of integer numbers.

Assumption 3.1. The cumulants \( k_{i_1, \ldots, i_p}(x) \) of the unknown risk-neutral density \( f_X \) are given.

Definition 3.4. Denote \( \psi(x, \xi) = \mathbb{E}[\exp(\xi |_{[i_1]} X_{[i_1]})] \) as the moment-generating function. The cumulant-generating function (CGF) of \( x \) is defined as:

\[ K(x, \xi) = \log \psi(x, \xi). \]

which is convergent for small \( \xi \).

This function can be expanded into the infinite series:

\[
\log \psi(x, \xi) = \xi_{[i_1]} k_{i_1}(x) + \xi_{[i_1,i_2]} k_{[i_1,i_2]}(x)/2! + \xi_{[i_1,i_2,i_3]} k_{[i_1,i_2,i_3]}(x)/3! + \cdots,
\]

which is convergent for small \( \xi \) where the terms \( k_{i_1, \ldots, i_p}(x) \) will be defined as the cumulants. The cumulant \( k_{i_1}(x) \) is the mean, \( k_{i_1,i_2}(x) \) is the variance, \( k_{i_1,i_2,i_3}(x) \) is a measure of skewness and \( k_{i_1,i_2,i_3,i_4}(x) \) is a measure of kurtosis. The expansion (11) can be used to find the values of \( k_{i_1, \ldots, i_p}(x) \).
Denote $M_{j_1,\ldots,j_p}$ as the difference of the moments of distributions $f_{X,g_S}$. In finance, the cumulants are commonly used. The covariance matrix is a cumulant of second-order. The difference of the moments can be expressed in terms of the difference of cumulants of $X$ and $S$ as:

\[
\begin{align*}
M_0 &= 1, \\
M_{l_1} &= k_{l_1}(x) - k_{l_1}(s), \\
M_{l_1,l_2} &= (k_{l_1,l_2}(x) - k_{l_1,l_2}(s)) + M_{l_1}M_{l_2}, \\
M_{l_1,l_2,l_3} &= (k_{l_1,l_2,l_3}(x) - k_{l_1,l_2,l_3}(s)) + (M_{l_1}(k_{l_2,l_3}(x) - k_{l_2,l_3}(s)) + M_{l_2}(k_{l_1,l_3}(x) - k_{l_1,l_3}(s))) + M_{l_3}(k_{l_1,l_2}(x) - k_{l_1,l_2}(s)) + M_{l_1}M_{l_2}M_{l_3}, \\
M_{l_1,l_2,l_3,l_4} &= (k_{l_1,l_2,l_3,l_4}(x) - k_{l_1,l_2,l_3,l_4}(s)) + (M_{l_1}(k_{l_2,l_3,l_4}(x) - k_{l_2,l_3,l_4}(s))) + (M_{l_2}(k_{l_1,l_3,l_4}(x) - k_{l_1,l_3,l_4}(s))) + (M_{l_3}(k_{l_1,l_2,l_4}(x) - k_{l_1,l_2,l_4}(s))) + (M_{l_4}(k_{l_1,l_2,l_3}(x) - k_{l_1,l_2,l_3}(s))) + M_{l_1}M_{l_2}M_{l_3}M_{l_4},
\end{align*}
\]

where,

\[
\begin{align*}
\{M_{l_1}(k_{l_2,l_3,l_4}(x) - k_{l_2,l_3,l_4}(s))\}^{(4)} &\equiv M_{l_1}(k_{l_2,l_3,l_4}(x) - k_{l_2,l_3,l_4}(s)) + M_{l_2}(k_{l_1,l_3,l_4}(x) - k_{l_1,l_3,l_4}(s)) + M_{l_3}(k_{l_1,l_2,l_4}(x) - k_{l_1,l_2,l_4}(s)) + M_{l_4}(k_{l_1,l_2,l_3}(x) - k_{l_1,l_2,l_3}(s)),
\end{align*}
\]

is the sum over the partitions of four indices in two. The binomial $^nC_2$ notation represents the four possible partitions of the set $\{l_1, l_2, l_3, l_4\}$ into two sets of one and three elements each.\(^8\) The notation,

\[
\{(k_{l_1,l_2}(x) - k_{l_1,l_2}(s))(k_{l_3,l_4}(x) - k_{l_3,l_4}(s))\}^{(3)} \equiv (k_{l_1,l_2}(x) - k_{l_1,l_2}(s))(k_{l_3,l_4}(x) - k_{l_3,l_4}(s)) + (k_{l_1,l_2}(x) - k_{l_1,l_2}(s))(k_{l_3,l_4}(x) - k_{l_3,l_4}(s)) + (k_{l_1,l_2}(x) - k_{l_1,l_2}(s))(k_{l_3,l_4}(x) - k_{l_3,l_4}(s)) + (k_{l_1,l_2}(x) - k_{l_1,l_2}(s))(k_{l_3,l_4}(x) - k_{l_3,l_4}(s)) + (k_{l_1,l_2}(x) - k_{l_1,l_2}(s))(k_{l_3,l_4}(x) - k_{l_3,l_4}(s)),
\]

represents the three different possible partitions of the set $\{l_1, l_2, l_3, l_4\}$ into two sets of two elements each. This number is equivalent to the number of partitions of the set of four elements into two sets, or the Stirling number $S_2(3) = 2^3 - 1 = 3$. Additional moments could be developed following combinatorics rules, and the work of McCullagh (1987) is a good reference for this purpose.

**Proposition 3.1.** Define $X$ as an $n$-variate stochastic process with a multivariate continuous density function $f_X$. Define $g_S$ to be another multivariate continuous distribution defined over the random vector $s$. This density will be the approximate density. Denote $m_{l_1,\ldots,l_p}(x)$ as the moment of order $p$ of $X$ and $k_{l_1,\ldots,l_p}(x)$ the cumulants of order $p$ of $X$. Then, the density $f_X$ can be expressed in terms of the following expansion:

\[
f_X = g_S + \sum_{j=1}^{n-1} M_{[1],[2],[\ldots],[j]} \frac{(-1)^j}{j!} \partial_s^{[1]} \partial_s^{[2]} \ldots \partial_s^{[j]} g_S + \varepsilon(s,n),
\]

where

\[
\varepsilon(s,n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(\imath \xi') \sigma(\|\xi\|^n) d\xi.
\]

This expansion will be termed the Multivariate Generalised Edgeworth Expansion (MGEE). The tensor notation $M_{[1],[2],[\ldots],[j]}$ refers to:

\[
M_{[1],[2],[\ldots],[j]} \equiv \sum_{l_1=1}^{n} \left( M_{l_1}(-1) \partial_{l_1} g_S + \sum_{l_2=1}^{n} \left( M_{l_1,l_2} \left( \frac{1}{2} \partial^2_{l_1,l_2} g_S + \ldots + \sum_{l_j=1}^{n} M_{l_1,\ldots,l_j} (-\frac{1}{j!} \partial_{l_1} \partial_{l_2} \ldots \partial_{l_j} g_S) \right) \right) \right)
\]

\(^8\)The binomial $^nC_2$ notation represents the possible partitions of the set $\{l_1, l_2, l_3, l_4\}$ into two sets of two elements each.
Proof. See Section A.1 of the appendix.

The MGEE approximation has been presented until the fourth-order moment. Continuous densities with their derivatives are candidates for the auxiliary distribution \( g_s \). Although the focus of this research is the approximation of option prices using multivariate densities, the MGEE is useful for any application where the density \( f_X \) is unknown, but the moments are available or they could be estimated.

4 MULTI-ASSET OPTION APPROXIMATION

The result presented in the previous section will be used to approximate the risk-neutral distribution \( f_{X(t)} \). It is sufficiently general that it can be used for different distribution approximations. However, if we want to use it for basket option pricing we will need to provide additional approximations. The density approximation is used towards finding the value of a European multi-asset option. A general case is presented for the arbitrary continuous-time price processes \( X(t) \) described in Section 2.1:

Corollary 4.1. Denote the \( n \)-variate continuous-time stochastic price process \( X(t) \) with a unique continuous density function \( f_{X(t)} \). Define \( S(t) \) as the multivariate lognormal process used to approximate \( X(t) \) and denote by \( g_S(t) \) the density function of \( S(t) \). Denote \( \Pi(S(t)) \) the payoff function, the value of an option \( C_t \) at time \( t = 0 \) can be approximated as:

\[
C_0(\Pi(X(t))) = \exp(-rt)\int_0^\infty \Pi(s(t))dG_S(t) + \exp(-rt)\sum_{j=1}^{n-1} M_{[i_1,...,i_j]} (-1)^j \frac{1}{j!} \int_0^\infty \Pi(s(t)) \frac{\partial^j}{\partial s_{[i_1]}(t) \cdots \partial s_{[i_j]}(t)} g_S(t)ds(t) + \varepsilon(\Pi(s(t)), n),
\]

where \( ds(t) = ds_1(t) \cdots ds_n(t) \) and,

\[
\varepsilon(\Pi(s(t)), n) = \frac{1}{2\pi} \int_0^\infty \exp(i\xi s(t))\alpha(\|\xi\|^n) d\xi.
\]

\( \Pi(s) \) will be equal to \( \Pi(X) \) substituting the components \( x_i \) by \( s_i \).

Proof. Using the risk-neutral pricing approach, the value of the option is:

\[
C_0(\Pi(X(t))) = \exp(-rt)\mathbb{E}_0^Q \left[ \Pi(X(t)) \middle| F_0 \right] = \exp(-rt)\int_0^\infty \Pi(s(t))dF_{X(t)}.
\]

Then the result follows immediately from applying Proposition 3.1.

Denote:

\[
C_{0,W}(\Pi(X(t))) = \exp(-rt)\int_0^\infty \Pi(s(t))dG_{s(t)},
\]

\[
\sum_{j=1}^{n-1} C_{0,W,[i_1,...,i_j]}(\Pi(X(t))) = \exp(-rt)\sum_{j=1}^{n-1} M_{[i_1,...,i_j]} (-1)^j \frac{1}{j!} \int_0^\infty \Pi(s(t)) \frac{\partial^j}{\partial s_{[i_1]}(t) \cdots \partial s_{[i_j]}(t)} g_S(t)ds(t),
\]

then the option price formula reveals three components:

\[
C_0(\Pi(X(t))) = C_{0,W}(\Pi(X(t))) + \sum_{j=1}^{n-1} C_{0,W,[i_1,...,i_j]}(\Pi(X(t))) + \varepsilon(\Pi(s(t)), n).
\]

The first part, \( C_{0,W}(\Pi(X(t))) \), is the value of the option under a simple Black and Scholes world, of a multivariate Wiener process with constant parameters, also known as geometric Brownian motion (GBM). In the univariate case this part will be reduced to a Black and Scholes formula (Jarrow and Rudd, 1982). Given that there still does not exist an equivalent Black and Scholes closed formula for the multivariate case, for numerical applications, or to calibrate the model, an approximation of the first section is required. There are very good
approximations for the bivariate case (spread options), including Borovkova et al. (2007), Li et al. (2008) and, for the multivariate case, Li et al. (2010) and Alexander and Venkatramanan (2012). However, due to the improved precision we use a Monte Carlo simulation method, integrating the payoff over the corresponding lognormal distribution.

The second part, $\sum_{j=1}^{n-1} C_{0,W}^{(n)}(\Pi(x(t)))$, is the correction given by the MGEE, or the difference between the moments of the asset distribution $f_X$ and the multivariate lognormal distribution $g_S$ times a partial derivative of the lognormal distribution. In the univariate case this second part will be reduced to a lognormal density. In the multivariate case it can be demonstrated that reducing these partial derivatives to a multivariate lognormal is equivalent to finding the density of the sum of lognormal distributions, and this problem is still unsolved. We derive expressions for the partial derivatives, and re-use the simulation paths of the first part of the formula to calculate the integrals.

The third component of the formula, $\varepsilon(\Pi(s(t)), n)$, is just the error of the approximation. We calculate some bounds for specific cases. These bounds will be determined in the section on the numerical efficiency of the model, for the case of an option defined over jump-diffusion processes.

4.1 Value of $C_{0,W}^{(n)}(\Pi(x(t)))$: fitting multivariate Wiener processes

The general structure of the formula to value options over multivariate arbitrary processes was outlined in (13). Before we can find a formula for specific cases to provide numerical applications, we must find an approximation of:

$$C_{0,W}^{(n)}(\Pi(x(t))) = \exp(-rt) \int_0^\infty \Pi(s(t)) dG_S(t).$$

(14)

Using as a payoff the definition of the basket option as in (2), the integral becomes,

$$\int_0^\infty [\omega_1 s_1(t) + \cdots + \omega_n s_n(t) - K]^+ dG_S(t).$$

This integral can be rewritten as:

$$\int [\omega_1 s_1(t) + \cdots + \omega_n s_n(t) - K] g_S(t) ds(t),$$

and this is just a function of the first moment of the multivariate density $g_S(t)$, truncated at the line $\omega_1 s_1(t) + \cdots + \omega_n s_n(t) \geq K$. We need to find the density of $g_S(t)$. It is straightforward to demonstrate that the density function $g_S(t)$ will be MVLN. Now, we find the parameters of $g_S(t)$:

Let the process $S(t)$ be defined as in (1), with initial values $S(0) = (S_1(0), \ldots, S_n(0))$. Define the vector $\log(S(t)) = (\log(S_1(t)), \ldots, \log(S_n(t)))$. Applying Itô’s lemma to each component $\log(S_i(t))$, and solving this differential equation we have:

$$\log(S_i(t)) = \log(S_i(0)) + \left( r - \frac{1}{2} \sigma_i^2 \right) t + \sigma_i W_i(t).$$

The distribution $g_S(t)$ will be $n$-variate lognormal with parameters:

$$\mu_s = \begin{pmatrix} \log(S_1(0)) + \left( r - \frac{1}{2} \sigma_1^2 \right) \mu \sigma \left( \begin{array}{c} \sigma_1^2 t \\ \sigma_2^2 t \\ \vdots \\ \sigma_n^2 t \end{array} \right) \\ \vdots \\ \log(S_n(0)) + \left( r - \frac{1}{2} \sigma_n^2 \right) \mu \sigma \left( \begin{array}{c} \sigma_1^2 t \\ \sigma_2^2 t \\ \vdots \\ \sigma_n^2 t \end{array} \right) \end{pmatrix},$$

$$\sigma_s = \sigma \left( \begin{array}{c} \sigma_1^2 t \\ \sigma_2^2 t \\ \vdots \\ \sigma_n^2 t \end{array} \right).$$

(15)

Define the vector $\log(S(t)) = (\log(S_1(t)), \ldots, \log(S_n(t)))$. Applying Itô’s lemma to each component $\log(S_i(t))$, and solving this differential equation we have:

$$\log(S_i(t)) = \log(S_i(0)) + \left( r - \frac{1}{2} \sigma_i^2 \right) dt + \sigma_i dW_i(t).$$

Having found the density parameters of $g_S(t)$, the problem of solving the integral (14) reduces to finding the first moment of the multivariate lognormal with parameters (15), truncated at the semi-plane $\omega_1 S_1(t) + \cdots +
\( \omega_n S_n(t) \geq K \). Assume that we have a bivariate case with payoff \( \Pi(S(t)) = [S_1(t) + S_2(t) - K]^+ \); the value of such option will be \( \mathbb{E}^Q_0 \left[ (S_1(t) + S_2(t) - K)^+ \right] \). In Figure 2a we have an example of a bivariate lognormal (BVln) distribution truncated at the line \([S_1(t) + S_2(t) - K]^+\), where the value of the option is the integral under the surface. The equivalent problem in the univariate case is to calculate the first moment of the univariate lognormal truncated at \( S_1(t) = 0.4 \): \( \mathbb{E}^Q_0 \left[ (S_1(t) - K)^+ \right] \) (see Figure 2b).

![Figure 2: The risk-neutral density of a GBM in the univariate an the bivariate case truncated at the payoff.](image)

The multivariate integral is approximated using a Monte Carlo method:

\[
C_{0,\mathcal{W}}(\Pi(x(t))) = \exp(-rt) \int \prod_{i=1}^{n} \omega_i s_i(t) \exp(\rho_i \sigma_i \sqrt{\tau} \phi_i^p - K) ds(t)
\]

\[
\approx \exp(-rt) \frac{1}{N} \sum_{p=1}^{N} \left( \sum_{i=1}^{n} \omega_i S_i(0) e^{(r - \frac{1}{2} \sigma_i^2) t + \sigma_i \sqrt{\tau} \phi_i^p - K} \right)^+,
\]

where \( N \) is the number of path simulations, \( n \) the number of assets, and \( \phi_i^p \) is a multivariate normal standard variable generated with correlations \( \rho_{i,j} \) between assets \( S_1, S_2 \) for the sample-path \( p \).

### 4.2 Value of \( C_{0,\mathcal{W},[t_1,...,t_j]}(\Pi(x(t))) \): corrections of the price by moments of the risk-neutral distribution

For the second part of the formula, the integral will be approximated with a Monte Carlo simulation, although other methods like the Laplace inverse transform are suggested for future extensions of this work for cases of low dimensionality.\(^9\) The sample-paths used to calculate the Wiener part \( C_{0,\mathcal{W}}(\Pi(x(t))) \) could be re-used to calculate the integrals of \( C_{0,\mathcal{W},[t_1,...,t_j]}(\Pi(x(t))) \). We proceed to calculate the partial derivatives:

It turns out that the partial derivatives are functions of the density \( g_S \):

\[
\frac{\partial^{\jmath}}{\partial S_{i_1} \cdots \partial S_{i_j}} g_S(t) = g_S(t) h \left( S_i(t), \ldots, S_j(t) \right),
\]

where \( h \left( S_i(t), \ldots, S_j(t) \right) \) is a function of \( S_i(t), \ldots, S_j(t) \). In Section A.2 of the appendix there is a detailed description of the form of the partial derivatives.

For calculating \( C_{0,\mathcal{W},[t_1,...,t_j]}(\Pi(x(t))) \), the moments \( M_{[t_1,...,t_j]} \) are given by the cumulants of the risk-neutral density \( k_{t_1,...,t_j}(x) \), and the cumulants \( k_{t_1,...,t_j}(s) \) of the MLN\((\mu, \Sigma)\) distribution are:

\[
k_{t_1,...,t_j}(s) = \mathbb{E} \left[ S_1^{\alpha_1} S_2^{\alpha_2} \cdots S_n^{\alpha_n} \right] = \exp \left( \frac{1}{2} \alpha^T \Sigma \alpha + \alpha^T \mu \right),
\]

\(^9\) As mentioned before, when the time index of the stochastic process is omitted we refer to the random variable at time \( t \): \( s \equiv s(t) \).
where \( j, \alpha_i = j \). The integrals are approximated using the Monte Carlo path simulations generated before for the calculation of \( C_{0,W}(\Pi(x(t))) \):

\[
\sum_{j=1}^{n-1} C_{0,W,[1,...,j]}(\Pi(x(t))) = \\
= \exp(-rt) \sum_{j=1}^{n-1} M_{[1,\ldots,[j],...]} \frac{(-1)^j}{j!} \int [\omega_1 s_1(t) + \ldots + \omega_n s_n(t) - K] \frac{\partial^j}{\partial s_{1t} \ldots \partial s_{jt}} g_{s(t)} ds(t) \\
\approx \exp(-rt) \left( \sum_{j=1}^{n-1} M_{[1,\ldots,[j],...]} \frac{(-1)^j}{j!} \right) \frac{1}{N} \sum_{p=1}^{N} h(s_1(t), \ldots, s_j(t)) \left( \sum_{i=1}^{n} \omega_i s_i(0) e^{(\mu_i - \frac{1}{2} \sigma_i^2) t + \sigma_i \sqrt{t} \Phi_i} - K \right).
\]

### 4.3 Analysis of the correction term \( C_{0,W,[1,...,j]}(\Pi(x(t))) \)

The term \( C_{0,W,[1,...,j]}(\Pi(x(t))) \) is developed further. With the intention of abbreviating the notation, the time parameter is omitted, therefore \( s(t) \equiv (S_1, \ldots, S_n) \). By definition, the density \( g_s \) is:

\[
g_s = (2\pi)^{-n/2} |\Sigma_s|^{-1/2} \left( \prod_{i=1}^{n} S_{i}^{-1} \right) \exp \left( -\frac{1}{2} (\log(s) - \mu_s)^\prime \Sigma_s^{-1} (\log(s) - \mu_s) \right), \tag{17}
\]

where \( \log(s) = (\log(S_1), \ldots, \log(S_n)) \) and \( \mu_s, \Sigma_s \) are the mean vector and covariance matrix defined in (15).

The second part of the option approximation (13) up to the second-order is:

\[
\sum_{j=1}^{n-1} C_{0,W,[1,...,j]}(\Pi(s(t))) = \exp(-rt) \sum_{t_1=1}^{n} M_{t_1} (-1) \int_0^{\infty} \Pi(s(t)) \frac{\partial}{\partial s_{1t}} g_s + \exp(-rt) \sum_{t_1=1}^{n} \sum_{t_2=1}^{n} M_{t_1,t_2} \frac{1}{2} \int_0^{\infty} \Pi(s(t)) \frac{\partial^2}{\partial s_{1t} \partial s_{2t}} g_s. \tag{18}
\]

Define by \( \Sigma_s^{-1} \) the inverse matrix of \( \Sigma_s \):

\[
\Sigma_s^{-1} = \left( \begin{array}{cccc}
\varsigma_{1,1} & \varsigma_{1,2} & \cdots \\
\varsigma_{2,1} & \varsigma_{2,2} & \cdots \\
\vdots & \vdots & \ddots 
\end{array} \right). \tag{19}
\]

After the analysis of the correction terms of \( C_{0,W,[1,...,j]}(\Pi(x(t))) \) up to the second-order, developed in the Section A.3 of the appendix, we have that the first-order moment correction of (18) becomes:

\[
\exp(-rt) \sum_{t_1=1}^{n} M_{t_1} (-1) \int_0^{\infty} \Pi(s(t)) \frac{\partial}{\partial s_{1t}} g_s = \exp(-rt) \left( \sum_{t_1=1}^{n} M_{t_1} (-1)(\Sigma_{s^{-1}})_{t_1} - 1 \right) \exp(-rt) \int_0^{\infty} \Pi(s(t)) g_s + \sum_{t_1=1}^{n} M_{t_1} (-1) \exp(-rt) \sum_{j=1}^{n} s_{1,t,j} \int_0^{\infty} \log(S_j) \Pi(s(t)) g_s.
\]

The term \( \log(S_j) = \log(S_j(t)) \) is a log-contract, and it is an essential instrument to hedge variance swaps, and moment swaps in general (see Neuberger, 1994; Demeterfi et al., 1999; Schoutens, 2005, for more details). In this case, the log-contract appears after applying the first-order partial derivative of the MVLN density, and it represents the sensitivity of the risk-neutral density to changes of \( S_{1t} \), and it could be considered a kind of ‘Delta’ of the risk-neutral density, with respect to future changes. It is related to the classical ‘Delta’ of changes with respect to the current stock price \( S_t(0) \). The log-contract is essential for variance swaps, and similarly seems to be essential for the sensitivity to changes of \( S_{1t} \).
The second-order moment correction of (18) is (see Section A.3 of the appendix):

\[
\exp(-rt) \sum_{l_1=1}^{n} M_{l_1,1} \frac{1}{2} \int_{0}^{\infty} \Pi(s(t)) \frac{\partial^2}{\partial S^2_{l_1}} g_S =
\]

\[
\exp(-rt) \left( \sum_{l_1=1}^{n} M_{l_1,1} \frac{1}{2} \exp(-2\mu_{l_1}) \right) \left( 2 - 3\Sigma_{l_1,1}^{-1} \mu_a + \left( \Sigma_{l_1,1}^{-1} \mu_a \right)^2 - \varsigma_{l_1,1} \right) \int_{0}^{\infty} \Pi(s(t)) g_S +
\]

\[
\sum_{l_1=1}^{n} M_{l_1,1} \frac{1}{2} \exp(-2\mu_{l_1}) \sum_{j=1}^{n} \left( \varsigma_{l_1,j} \right)^2 \int_{0}^{\infty} \Pi(s(t)) \log(S_j) g_S +
\]

and the second-order cross-moment correction is:

\[
\exp(-rt) \sum_{l_1=1}^{n} \sum_{l_2=1}^{n} M_{l_1,l_2} \frac{1}{2} \int_{0}^{\infty} \Pi(s(t)) \frac{\partial^2}{\partial S^2_{l_1}} g_S = \exp(-rt)
\]

\[
\left( \sum_{l_1=1}^{n} \sum_{l_2=1}^{n} M_{l_1,l_2} \frac{1}{2} \exp(-\mu_{l_1} - \mu_{l_2}) \left( 1 - \Sigma_{l_1,l_2}^{-1} \mu_a - \Sigma_{l_1,l_2}^{-1} \mu_a + \Sigma_{l_1,l_2}^{-1} \mu_a \Sigma_{l_1,l_2}^{-1} \mu_a - \varsigma_{l_1,l_2} \right) \int_{0}^{\infty} \Pi(s(t)) g_S +
\]

\[
\sum_{l_1=1}^{n} \sum_{l_2=1}^{n} M_{l_1,l_2} \frac{1}{2} \exp(-\mu_{l_1} - \mu_{l_2}) \sum_{j=1}^{n} \left( \varsigma_{l_1,j} \right)^2 + \varsigma_{l_2,j} \left( 1 - \Sigma_{l_1,l_2}^{-1} \mu_a \right) \right) \int_{0}^{\infty} \Pi(s(t)) \log(S_j) g_S +
\]

\[
\sum_{l_1=1}^{n} \sum_{l_2=1}^{n} M_{l_1,l_2} \frac{1}{2} \exp(-\mu_{l_1} - \mu_{l_2}) \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \left( \varsigma_{l_1,j_1} \varsigma_{l_2,j_2} \right) \int_{0}^{\infty} \Pi(s(t)) \log(S_{j_1}) \log(S_{j_2}) g_S.
\]

In this case we have calculated second-order sensitivities of the risk-neutral density to changes of \(S_j^2(t)\), or a ‘Gamma’ equivalent of the risk-neutral density. The quadratic log-contract functions will produce quadratic volatility terms or variance terms, essentials for calculating the sensitivity of the risk-neutral density to \(S_j^2(t)\).

We could extrapolate that the correction terms of higher-order consist of:

1. Functions of Wiener processes related to \(C_{0,W}(\Pi(x(t)))\) with transformed MVLN densities,
2. Sums of log-contracts times Wiener processes,
3. Sums of cross log-contracts of higher-order times Wiener processes.

The option price approximation could be separated in three terms as in (13). The first term, \(C_{0,W}(\Pi(x(t)))\), is an integral over a MVLN density as in the Black and Scholes (1973) world. For the second term, \(C_{0,W}[t_1,\ldots,t_j](\Pi(x(t)))\), we analysed the expansion up to the second-order and we found that it could be expressed as an integral of shifted MVLN densities times log-contracts and cross log-contracts. The expansions beyond the second-order will have a similar pattern given the nature of the MVLN density. This result could be used to hedge the risk-neutral density using the moments of higher-order, and further important theories could be developed from this result.

### 4.4 Analysis of the error term \(\varepsilon(\Pi(s(t)), n)\)

We note that the error term of the MGEE is:

\[
\varepsilon(\Pi(s(t)), n) = \frac{1}{2\pi} \int_{0}^{\infty} \exp(i\xi's(t))o(\|\xi\|^n)d\xi = \sum_{j=n}^{\infty} M_{[1,\ldots,|j|,\ldots]} \frac{(-1)^j}{j!} \frac{\partial^j}{\partial S_{1} \cdots \partial S_{j}} g_S.
\]

An analysis of this error term for arbitrary distributions is beyond the scope of this research. Numerical analyses in the univariate GEE for the lognormal case were done by Schleher (1977) and by Jarrow and Rudd (1982), and a deeper analysis of multivariate Edgeworth expansions was done by Skovgaard (1986). In our case, all the cumulants of the MVLN \(k_{l_1,\ldots,l_j}(s)\) exist (see Equation 16), and in case all cumulants of the risk-neutral density
$k_{t_1, \ldots, t_j}(x)$ exist the difference of cumulants $M_{t_1, \ldots, t_j}$ will be finite. Then, assuming,

$$\lim_{n \to \infty} M_{[t_1, \ldots, t_j]} \frac{(-1)^j}{j!} = 0.$$  

it can be shown that,

$$\lim_{n \to \infty} \sup \| \epsilon(\Pi(s(t)), n) \| = 0,$$

noting by the result of previous section that,

$$\lim_{n \to \infty} \frac{\partial^n}{\partial s_{t_1} \ldots \partial s_{t_n}} g_S \approx \lim_{n \to \infty} \frac{\log^n(S_{t_j})}{S_{t_j}^n} g_S = 0.$$  

### 4.5 Performance of the MGEE

The MGEE requires an approximating distribution to fit the risk-neutral density. For pricing multi-asset options like spread or basket options, we will need to numerically integrate the expected payoff as it does not exist a closed-form solution for these particular cases. Analytical approximations like Li et al. (2010) were tested for the expansion of the correction terms in Section 4.3; however, results showed that the errors of the closed-form approximation were amplified when they were used with the MGEE, and precision is the most important feature of the expansion. A solution is to generate Monte Carlo sample-paths for pricing all the terms of the expansion. To select an appropriate number of simulations paths, we evaluated time and precision. The precision of the Monte Carlo algorithm increases at a rate of $1/\sqrt{N}$ for any dimension, a favourable attribute for high-dimension problems. In Figure 3 we plot the standard deviation of the Monte Carlo integration for the increasing number of paths. Valuation of the integral was tested for pricing two different multi-asset options with jump-diffusion defined in Section 5.2. Valuations with up to 50,000,000 simulations were tested, finding that 20,000,000 simulations would provide an approximate value with an error of approximately $0.3\%$ for jump-diffusions with an intensity of $\lambda = 1$, and of approximately $10\%$ for jump-diffusions with an intensity of $\lambda = 10$. In Table 1, we have the running time of the Monte Carlo algorithm, and the additional time consumed by the successive MGEE. The second-order MGEE will consume only $30\%$ more time than the MC algorithm, while the fourth-order MGEE will consume approximately eight times the time consumed by the Monte Carlo algorithm. Considering, that the option price precision could be improved in some cases by $20\%$ (see Table 5), the MC option pricing with MGEE for cases when moments of the risk-neutral density are available would be a suitable decision.
Table 1: Algorithm performance when moments of higher-order are included in the MGEE approximations of jump-diffusion processes for different calibration methods. The columns MGEE2, MGEE3, and MGEE4 of the rows Uncalibrated, \( h_2(\hat{\sigma}) \), \( h_3(\hat{\sigma}) \), \( h_4(\hat{\sigma}) \) are the average running time of the option pricing and calibration algorithms for the 48 cases of Tables 9, 12, 13 and 14, respectively. The column ‘MC’ is the average running time of the Monte Carlo algorithm with 20,000,000 simulations.

<table>
<thead>
<tr>
<th>Objective Function</th>
<th>MC</th>
<th>MGEE2</th>
<th>MGEE3</th>
<th>MGEE4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uncalibrated</td>
<td>581</td>
<td>172</td>
<td>591</td>
<td>3760</td>
</tr>
<tr>
<td>( h_2(\hat{\sigma}) ) = |M_{1,1}|_2^2</td>
<td>581</td>
<td>177</td>
<td>603</td>
<td>3750</td>
</tr>
<tr>
<td>( h_3(\hat{\sigma}) ) = |M_{1,1}|<em>2^2 + |M</em>{1,1}|<em>2^2 + |M</em>{1,1}|_2^2</td>
<td>581</td>
<td>183</td>
<td>631</td>
<td>3925</td>
</tr>
<tr>
<td>( h_4(\hat{\sigma}) ) = |M_{1,1}|<em>2^2 + |M</em>{1,1}|<em>2^2 + |M</em>{1,1}|<em>2^2 + |M</em>{1,1}|_2^2</td>
<td>581</td>
<td>185</td>
<td>652</td>
<td>4210</td>
</tr>
</tbody>
</table>

4.6 Empirical motivation of moments difference

The partial derivatives and the cumulants’ corrections \( C_{X,t} \) in (13) have an economic sense. Denote the three correction terms:

\[
M_{c,2} = M_{[t_1,t_2]} \left( -\frac{1}{2!} \right) \frac{\partial^2}{\partial s_{t_1} \partial s_{t_2}} g_{t} \gamma_{t} \text{ (Second)},
\]

\[
M_{c,3} = M_{[t_1,t_2,t_3]} \left( -\frac{1}{3!} \right) \frac{\partial^3}{\partial s_{t_1} \partial s_{t_2} \partial s_{t_3}} g_{t} \gamma_{t} \text{ (Third)},
\]

\[
M_{c,4} = M_{[t_1,t_2,t_3,t_4]} \left( -\frac{1}{4!} \right) \frac{\partial^4}{\partial s_{t_1} \partial s_{t_2} \partial s_{t_3} \partial s_{t_4}} g_{t} \gamma_{t} \text{ (Fourth)}.\]

The value \( M_{c,2} \) represents the density variance correction. If the variance of \( f_{X,t} \) is greater than the variance of \( g_{t} \), the option price premium of the second cumulant is positive. Volatility has a positive premium. The value \( M_{c,3} \) is the density skewness correction. If the skewness of \( f_{X,t} \) is greater than the skewness of \( g_{t} \), the option premium is negative; skewness is transmitted into the option price as a negative correction. The value \( M_{c,2} \) is the kurtosis correction. If the kurtosis of \( f_{X,t} \) is greater than the kurtosis of \( g_{t} \), the option premium is positive; kurtosis is transmitted into the option price as a positive correction. All these relations are consistent with Jarrow and Rudd’s (1982) results, except for the skewness correction, which differs because we have included the \( \tau_j \) term. In Rubinstein (1998) the relationship between the univariate moments of higher-order and the prices of options are modelled in binomial trees. This research is useful as an extension for our work.

Figures 4a, 4b and 4c plot the moment corrections for two examples: a bivariate jump-diffusion process defined in Section 4.7 with symmetrical volatilities \( \sigma_1 = \sigma_2 = 0.2 \), and a second with asymmetrical volatilities \( \sigma_1 = 0.1, \sigma_2 = 0.26458 \). Figure 4 shows that options where the difference of volatilities between the assets are higher will have a greater impact in all moments’ differences \( M_{1,1,⋯,t} \in \{1,\ldots,4\} \). Figure 4 also displays important information on the partial derivatives that could be used for sensitivity analysis in changes of the risk-neutral measures. Figure 4 shows the second, third and fourth cumulant differences total corrections (the sum of the cross-moments of the \( n \)-th-order correction). The second cumulant difference correction has one inflection point near the at-the-money (ATM) price, the third cumulant difference correction has two inflection points, one in-the-money (ITM) and the other out-of-the-money (OTM), and the fourth cumulant difference correction has three inflection points in ITM, ATM and OTM prices. This is consistent with the results of Jarrow and Rudd (1982). Figure 4 shows two cases: one with the components of volatility equal \( \sigma_j = 0.2 \) in a straight line, and other where the component volatilities are different in a dashed-dot line. When the univariate density of the payoff is considered, the total volatility in both cases is equal, but when we consider a multivariate density a greater difference in the component volatilities will be transmitted as greater corrections in the price of the option by the moments.

4.7 Analysis of the density values of the MGEE approximation

The selection of the MVLN as an auxiliary density comes as a result of two important attributes: it is the density on which the GBM process converges, the one which is widely used in the industry and is still the reference for option pricing in academia, and it possesses heavy tails.

The second attribute is of great importance when we need to validate that the resulting density approximation is in fact a density. The main problem that is faced using the MGEE to approximate a pdf is the possibility of
Figure 4: Second-, third- and fourth-order corrections with the partial derivatives of the MGEE approximation of two bivariate jump-diffusion processes. Both processes have $\lambda = 0.5, \nu_i = 0.1, \delta_i = 0, \rho_{i,j} = 0, r = 0.05, t = 0.25, S_i(0) = 30, i, j \in \{1, 2\}$. The solid line is the total moment correction of the first process with $\sigma_1 = \sigma_2 = 0.2$, and the dash-dot line is the second process with $\sigma_1 = 0.1, \sigma_2 = 0.26458$. 

(a) Second cumulant difference total correction.

(b) Third cumulant difference total correction.

(c) Fourth cumulant difference total correction.
negative values in certain domains of the function approximation. The MGEE is a polynomial expansion over the auxiliary density, where the parameters of the polynomial are $M_{(t_1,\ldots,t_j)} [-(s_1^2 + \sigma_2^2)/2]$. If the multiplication of these parameters is negative enough to outweigh the positive part of the function, the resulting MGEE function approximation will be negative.

Let us consider the bivariate density of a jump-diffusion process with the parameters outlined in Example 1 of Section 2.3. Now iterate over different jump intensities $\lambda = 0.1, 1, 10$. Increasing the intensity of the jumps will increase the higher-order moments, therefore the skewness and the kurtosis and $M_{(t_1,\ldots,t_j)}$ will be higher. Figures 5a–5f plot the MGEE produce over a MVLN bivariate density to fit the desired moments. In the extreme cases of $\lambda = 1$ and $\lambda = 5$, the resulting MGEE density has negative values.

To overcome this problem, the selection of an auxiliary function with heavy tails will be a determining factor in the success of the application of the method. A calibration methodology to decrease the values of $M_{(t_1,\ldots,t_j)}$ will be developed in Section 6.

Example 2. The jump-diffusion process parameters for the numerical example are $\lambda = 0.5, \nu = 0.1, \delta = 0, S_1(0) = S_2(0) = 30, r = 0.05, t = 1$. Two volatility scenarios are generated:

1. In the first scenario, the risk-neutral density denoted by $f_1(X(t))$ has equal diffusion volatility for both assets, $\sigma_1 = \sigma_2 = 0.2, \sqrt{\sigma_1^2 + \sigma_2^2} = 0.2828$.
2. In the second scenario, the risk-neutral density denoted by $f_2(X(t))$ has different diffusion volatility, $\sigma_1 = 0.1, \sigma_2 = 0.26458, \sqrt{\sigma_1^2 + \sigma_2^2} = 0.2828$.

The auxiliary densities used by the MGEE are denoted by $g_1(S_1(t)), g_2(S_2(t))$. They have MVLN($\mu_1, \Sigma_1$), and MVLN($\mu_2, \Sigma_2$) distributions, and the same parameters $S_1(0), r, t$ of $f_1(X(t))$ and $f_2(X(t))$, respectively for $\mu_1, \mu_2, \Sigma_1, \Sigma_2$ as in (15). The value of a basket option with payoff $\Pi(S(t)) = (S_1(t) + S_2(t) - K)^+ \quad \text{with} \quad K = 60$, is calculated using a Monte Carlo simulation as described in Section 4.

In Table 2, option prices derived from the application of the MGEE approximation with different levels of moments are exhibited. The J-D Price column is the Monte Carlo option price of the contract with a jump-diffusion process, with 20,000,000 sample-path simulations. The Wiener column represents the option price without jump-diffusion. Subsequent columns $MGEE2, MGEE3,$ and $MGEE4$ are the option prices approximated including the second-, third- and fourth-order cumulant corrections in the polynomial expansion. The %pdf column is the percentage of the simulated sample-paths that when the MGEE is applied generate a positive density function, and the asterisk denotes the best approximation. By arbitrage arguments, the first cumulants of $f_1(X(t))$ and $g_2(S_1(t))$ are equal, as in the case of $f_2(X(t))$ and $g_2(S_1(t))$.

Let us assume the current market risk-neutral density is $f_1(X(t))$ and the next time available for hedging is $f_2(X(t))$. Remember that the rebalancing of the portfolio for hedging in real applications is possible only on a finite number of times. The amount of premium for the option price is symmetrical in case the volatility is symmetrical; However, the premium shifts towards the second asset for $f_2(X(t))$. This is expected as the premium values the difference $f_2(X(t)) - g_2(S_1(t))$. The reduction of the asset’s volatility $\sigma_2$ from 0.2 to 0.1 is reflected as an increase in the cumulant’s difference of 0.158, and this represents an 0.01 (0.19%) positive premium in the option price. If we use the univariate density of the sum of lognormals, as in Jarrow and Rudd (1982), an increase of the second- and fourth-order cumulants is reflected as a positive premium, while a decrease in the third-order cumulant is considered as a negative premium. In the multivariate case the cross-cumulants cause an additional effect. Although the fourth cumulant differences of $f_2(X(t)) - g_2(S_1(t))$ are positive, the asymmetry generates a total negative premium of $-0.0586$ or $-1.14\%$. A risk manager can use this information to evaluate the price impact when the risk-neutral density cumulants evolve over the time. To measure if there is any negative section of the MGEE density function, we simulate asset prices $(S_1(t), S_2(t))$ over the function domain, and calculate the associated density function. After 20,000,000 paths simulations, in both approximations there were no negative density points found.

---

10Das and Uppal (2004) found through empirical study that $\lambda$ in high-beta emerging markets is lower than 0.1.
11In a complete market there exist only one risk-neutral density. However, we could be interested in examining the differences between the risk-neutral densities of two different markets; for example, the risk-neutral density of a stock with the risk-neutral density of a commodity.
Figure 5: Effects over the MGEE density when the difference of the moments, $\|M_{l_1,\ldots,l_j}\|$, is increased, for a bivariate jump-diffusion process with $\lambda = (0.1, 1, 5), \nu_i = 0.1, \delta_i = 0, \sigma_i = 0.2, \rho_{i,j} = 0, r = 0.05, i, j \in \{1, 2\}, S_i(0) = 30$ fitted using a MVLN($\mu, \Sigma$), $\mu_i = \log(s_i(0)) + (r - \frac{1}{2}\sigma_i^2)t, \Sigma_{i,i} = \sigma_i t$ with the same parameters $S_i(0), \sigma_i, r, t$. 

(a) MGEE density with $\lambda = 0.1$.

(b) MGEE heatmap with $\lambda = 0.1$.

(c) MGEE density with $\lambda = 1$.

(d) MGEE heatmap with $\lambda = 1$.

(e) MGEE density with $\lambda = 5$.

(f) MGEE heatmap with $\lambda = 5$. 


Table 2: Option prices of a basket with payoff $\Pi(S(t)) = (S_1(t) + S_2(t) - K)^+$ when $S_1(t)$ and $S_2(t)$ are jump-diffusion processes with common parameters $\lambda = 0.5, \nu = 0.1, \delta = 0, S_1(0) = S_2(0) = 30$ and the basket has parameters $r = 0.05, t = 1$. We have two examples, one where the volatility is $\sigma_{f_1, x(t)} = (0.2, 0.2)$ and other where the volatility is $\sigma_{f_2, x(t)} = (0.1, 0.26458)$ with risk-neutral densities $f_{1, x(t)}$ and $f_{2, x(t)}$, respectively. The option prices were approximated applying a MGEE over the auxiliary $g_{i, x(t)}$ MVLN densities ($i = 1, 2$) without any calibration of the density parameters.

<table>
<thead>
<tr>
<th></th>
<th>J-D Price</th>
<th>MC Wiener</th>
<th>MGEE2</th>
<th>MGEE3</th>
<th>MGEE4</th>
<th>%pdf</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{1, x(t)}; g_{1, x(t)}$</td>
<td>5.1668</td>
<td>4.9780</td>
<td>5.2113</td>
<td>5.2014</td>
<td>5.1685*</td>
<td>1.00</td>
</tr>
<tr>
<td>$f_{2, x(t)}; g_{2, x(t)}$</td>
<td>5.1114</td>
<td>4.9292</td>
<td>5.1761</td>
<td>5.1714</td>
<td>5.1127*</td>
<td>1.00</td>
</tr>
</tbody>
</table>

Table 3: Option prices of a basket with payoff $\Pi(S(t)) = (S_1(t) + S_2(t) - K)^+$ when $S_1(t)$ and $S_2(t)$ are jump-diffusion processes with common parameters $\lambda = 0.5, \nu = 0.1, \delta = 0, S_1(0) = S_2(0) = 30$ and the basket has parameters $r = 0.05, t = 1$. We have two examples, one where the volatility is $\sigma_{f_1, x(t)} = (0.2, 0.2)$ and other where the volatility is $\sigma_{f_2, x(t)} = (0.1, 0.26458)$ with risk-neutral densities $f_{1, x(t)}$ and $f_{2, x(t)}$, respectively. The option prices were approximated applying a MGEE over the auxiliary $g_{i, x(t)}$ MVLN densities ($i = 1, 2$) without any calibration of the density parameters.

<table>
<thead>
<tr>
<th></th>
<th>J-D Price</th>
<th>MC Wiener</th>
<th>MGEE2</th>
<th>MGEE3</th>
<th>MGEE4</th>
<th>%pdf</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_{1, x(t)}; g_{1, x(t)}$</td>
<td>5.1668</td>
<td>4.9778</td>
<td>5.2145</td>
<td>5.2002</td>
<td>5.1505*</td>
<td>0.9999</td>
</tr>
<tr>
<td>$f_{2, x(t)}; g_{2, x(t)}$</td>
<td>5.1114</td>
<td>4.9184</td>
<td>5.1643</td>
<td>5.1621</td>
<td>5.0802*</td>
<td>0.9505</td>
</tr>
</tbody>
</table>

Figure 6: Norm projection of the MGEE density of $||S_1(t), S_2(t)||$ when a jump-diffusion process with $\lambda = 0.5, \nu = 0.1, \delta = 0, \sigma_1 = \sigma_2 = 0.2, \sqrt{\sigma_1^2 + \sigma_2^2} = 0.2828, S_1(0) = S_2(0) = 30, r = 0.05, t = 1$ is fitted (99.99% positive function).

5 Numerical Analysis of Multi-Asset Option Pricing: Methods Comparison

Consider that we are in a jump-diffusion risk-neutral world as in Merton (1976), but an asset manager does not acknowledge the presence of jumps, and actually he prices the options in the market considering only the Wiener diffusions (GBM). The mispricing will be related to the size of volatility and the drift of the jumps, but let us assume both are unknown for the asset manager. In this section we developed a set of numerical examples to test the benefits of measuring risk-neutral moments and using a MGEE, against using classical multi-asset options that do not incorporate this information. The analytic approximations of Li et al. (2010) and Alexander and Venkatraman (2012) were developed and used to compare with the results of the MGEE, additionally to the results of plain vanilla Monte Carlo methodology. In Section 6, we will improve this test incorporating the risk-neutral moments information for all four different methodologies.
5.1 Multivariate Merton’s jump-diffusion

To measure the option pricing corrections with a practical example, we select as the candidate for the risk-neutral density to be approximated $f_X(t)$, the density on which a jump-diffusion (J-D) process of Merton (1976) converges. This process will also be used to measure the numerical efficiency of the MGEE option pricing approximation in the Section 6. We extend the definition of Merton processes to the multivariate case:

Definition 5.1. Denote the multi-asset jump-diffusion (MJ-D) to the n-variate stochastic process $\mathbf{X} = \{X_i(t) \in \mathbb{R}^+, t \geq 0\}, i \in \{1, \ldots, n\}$, described by:

$$dS_i(t) = \mu_i dS_i(t) dt + \sigma_i X_i(t)dW_i(t) + (J_i(t) - 1)dP_i(\lambda),$$

where $W_i(t)$ are Wiener processes, $P_i(\lambda)$ is a Poisson process with intensity parameter $\lambda$, and $(J_i(t) - 1)$ represents the jump-size. The jump size has a normal distribution: $J_i(t) \sim \phi (\delta, \nu^2)$. We assume that the jump’s size and the jump’s occurrence are independent, therefore uncorrelated between,

$$\langle J_i(t), J_j(t) \rangle = \langle dP_i(t), dP_j(t) \rangle = 0,$$

with $i, j \in \{1, \ldots, n\}, i \neq j$, likewise the Wiener processes and the jumps:

$$\langle dW_i(t), dP_j(t) \rangle = \langle dP_i(t) = J_j(t) \rangle = 0.$$

On average, the MJ-D will be similar to a GBM diffusion:

$$dS_i(t) = \mu_i S_i(t) dt + \sigma_i S_i(t)dW_i(t),$$

but every $\lambda$ times it jumps $J_i(t) - 1$, generating the change in the asset $i$:

$$dS_i(t) = \mu_i S_i(t) dt + \sigma_i S_i(t)dW_i(t) + (J_i(t) - 1).$$

For this process to be a martingale, the drift needs to be extracted:

$$dX_i(t) = \left( r - \frac{1}{2} \sigma_i^2 - b \right) X_i(t) dt + \sigma_i X_i(t)dW_i(t) + (J_i(t) - 1)dP_i(\lambda),$$

where $r$ is the constant risk-free interest rate and $b$ is the adjustment due to the jump process. Merton found that, if the jumps are i.i.d., the price process will be lognormal distributed. In the multivariate case, $\mathbf{X}(t)$ will have a MVLN($\mu, \Sigma$) distribution. Applying the results of Das and Uppal (2004)\textsuperscript{12} to the moments of $dX_i(t)/X_i(t)$, we calculate the values of the parameters of $\mu, \Sigma$:

$$\mu = \begin{pmatrix} \log(S_i(0)) + (r - \frac{1}{2}(\sigma_i^2 + \lambda(\delta_i^2 + \nu_i^2)))t \\ \vdots \\ \log(S_n(0)) + (r - \frac{1}{2}(\sigma_n^2 + \lambda(\delta_n^2 + \nu_n^2)))t \end{pmatrix}, \quad \Sigma = \begin{pmatrix} (\sigma_1^2 + \lambda(\delta_1^2 + \nu_1^2))t & \cdots \\ \vdots & \ddots \end{pmatrix}.$$

Consequently, the value for $b$ that transforms the density of the process in a martingale is:

$$b = \frac{1}{2} \lambda (\delta_i^2 + \nu_i^2) t.$$

Merton established that $\delta_i$ must be equal to zero for the drift of the process to be zero.\textsuperscript{13} In Bates (1991) additional expressions for $\delta_i$ are derived where $\delta_i \neq 0$ to generate asymmetric jump-diffusion processes.

\textsuperscript{12}In Das and Uppal (2004), the moments of the multivariate returns $dX_i(t)/X_i(t)$ are calculated with the characteristic function. Das and Uppal assume a perfect correlation between the jumps: $\langle J_i(t), J_j(t) \rangle = 1$. In our case to simplify results we assume independent jumps, but jumps’ correlations different to 0 and 1 can easily be modelled with the characteristic function.

\textsuperscript{13}If $\delta_i$ is not zero, the jump-diffusion price process changes $dX_i(t)/X_i(t)$ are not MVN, and the third- and fourth-order cumulants are:

$$k_{1,2,3,4} = \lambda (\mu_1 \mu_2 \mu_3 + \mu_1 \mu_2 \mu_4 + \mu_1 \mu_2 \mu_5 + \mu_1 \mu_2 \mu_6),$$

for $i_1, \ldots, i_4 \in \{1, \ldots, n\}$. 

20
The cumulants of \( f_{X(t)} \) will be necessary to calculate the option price. In the univariate case, a closed-form density for \( X(t) \) is provided by Merton. For the MVLN moments we use the expression in (16). The first four cumulants are calculated using the expressions in (12), clearing the \( k_{1,\ldots,j} \) variables.

Despite the fact that we have a closed-form expression for \( f_{X(t)} \), we use a MGEE to approximate the option price. The auxiliary function \( g_{S(t)} \) will be a MVLN(\( \bar{\mu}, \bar{\Sigma} \)) similar to \( f_{X(t)} \), with the total volatility of the assets without the jump effect (\( \delta_i = 0, \nu_i = 0 \)):

\[
\begin{align*}
\hat{\sigma}_i &= \sigma_i + \lambda (\nu^2), \\
\bar{\sigma}_i &= \sigma_i,
\end{align*}
\]

where \( \hat{\sigma}_i \) is the total volatility of the jump-diffusion assets, and \( \bar{\sigma}_i \) is the total volatility of the simple diffusion assets. The parameters of the simple diffusion are the same as (15).

### 5.2 Pricing basket options over multivariate jump-diffusion processes

Numerical results for pricing basket options are presented in Table 9 in Appendix B, where the risk-neutral density \( f_{X(t)} \) is generated by a 5-dimensional jump-diffusion process with parameters \( \lambda \in [1, 10], \delta_i = 0, \nu_i \in \{0.05, 0.20\}, r \in [0.05, 0.10], t \in [0.25, 1], \sigma_i = 0.2, (S_i(0), \ldots, S_i(0)) = (35, 25, 20, 15, 5), i \in [1, \ldots, 5] \). The payoff of the basket option to be calculated is \( \Pi(S(t)) = \left( \sum_{i=1}^{5} S_i(t) - K \right)^+ \) with \( K \in \{90, 100, 110\} \). We focus our attention not only on the precision of the MGEE approximation, but on the contribution of the differences in the cumulants of different risk-neutral density states. The columns AV2012 and Li2010 represent the option price of Alexander and Venkatramanan (2012) and Li et al. (2010) methodologies.

Consider a situation where the real market evolves either by Wiener states or J-D states. We could estimate \( \lambda, \nu \) as in Das and Uppal (2004), but we would have no information about the impact of risk-neutral moments on the price. Additional hedging strategies could be generated with this information. For a risk manager, the differences in the prices between the Wiener and J-D columns are the price premium caused by the jumps. The increase of \( \lambda \) and \( \nu \) will increase the difference between these columns. In the options deep OTM, the price difference is even higher, as an effect of the higher cumulants caused by the jumps, and the wider region on which the payoff will be positive for J-D. Despite the higher cumulants, the price difference of the Wiener and J-D columns for options deep ITM is small, caused by the narrower region over which the payoff will be zero.

In comparison with AV2012 and Li2010, the MGEE produces better results, given the fact that it acknowledges the information of the jump-diffusion states through the moment of the risk-neutral diffusion (see Table 4 and Table 5). Second-order price corrections are the most important in OTM options (16 cases). They reduce the absolute difference in pricing between Wiener and J-D from 24.41% to 15.31% on average considering all cases, and from 14.02% to 5.06% when processes with a lower jump-intensity \( (\lambda = 1) \) are selected. Additionally, in 31.25% (5 of 16) of the cases MGEE2 is the best approximation. The increase of the volatility will increase the cumulants of the risk-neutral density, hence the option price will be higher. This is a well-known fact in finance, but it is now possible to have a measure of the impact of the higher-order moments of the risk-neutral density over the price option with a systematic approach.

Third- and fourth-order corrections add noise in the extreme case of \( \lambda = 10 \) and \( \lambda = 0.20 \). Nevertheless, these values are extreme as the reported values in Das and Uppal (2004) with real market data, which were in the range of \( \lambda \in (0.0138, 0.0501) \) and \( \nu \in (0.0792, 0.1185) \) for equity indices of developed countries. Equity indices of emerging markets report a higher jump-volatility \( \nu \), but the jump intensity \( \lambda \) is still much lower than the parameters considered in these examples, and also the multiplication of the jump volatility by the intensity is much lower in the examples considered. For the cases with a lower jump-intensity \( (\lambda = 1) \), the third- and fourth-order corrections reduce the absolute price difference of Wiener and J-D from 14.02% to 4.26% and 13.74%, respectively.

### 5.3 Contribution of the MGEE for option pricing

The MGEE approximation can be used in asset pricing theory for two main purposes: hedging and option pricing. In this section we focused on the former and we develop a calibration algorithm in Section 6 for the latter application. In Section 2 we demonstrated an example where the use of the univariate density of the sum of \( n \) asset prices for hedging will lead to a misspecification of the optimal hedge. Even a proper multivariate hedging model can mislead a risk manager if there is no change in the price of the option, but the portfolio weights must be adjusted using different Deltas for every asset in the basket.
A risk manager can use the moments of the risk-neutral density to detect the sources of risk of a multi-asset option. An MGEE approximation can be used to measure the price premium associated with an increase or decrease in certain moments of the risk-neutral density.

6 CALIBRATION AND NUMERICAL EFFICIENCY OF THE APPROXIMATION

In this section we measure the precision and the efficacy of the MGEE approximation, and we compare it with the other three different option pricing methodologies: plain vanilla Monte Carlo, Li et al. (2010) and Alexander and Venkatramanan (2012). This time we developed a test where the four different processes acknowledge the information of the moments of the risk-neutral density. The effects of higher-order moments for Wiener-based algorithms (Monte Carlo; Li et al., 2010; Alexander and Venkatramanan, 2012) will be contained in the optimisation of the volatility. In Zhao et al. (2013) it is mentioned that the effect of skewness and kurtosis of the risk-neutral density is incorporated in the volatility structure. A main concern in Section 5 was the possibility of negative MGEE density values, and their effect over the precision of the algorithm. The precision of the expansion depends on the difference of cumulants against the selected density. If the application is to hedge a risk-neutral density $f_{X(t)}$ with another density $g_{S(t)}$, the selection of the auxiliary density is based on the future scenarios; and the extent to which $g_{S(t)}$ can be adjusted to $f_{X(t)}$ will be limited to the constraints of the risk model. Generally, large deviations from $f_{X(t)}$ are the typical scenarios to be tested. But when we apply MGEE to price an option, we can select $g_{S(t)}$ and distort its moments to fit $f_{X(t)}$ over most of its domain. The calibration algorithm reduces the difference of the cumulants of $f_{X(t)}$ and $g_{S(t)}$.

6.1 Calibration algorithm

Given an unknown density $f_{X(t)}$ with known moments or cumulants, over which a payoff $\Pi(x(t))$ is defined, the objective is to select a MVLN($\mu, \Sigma$) density with moments that are close as possible to the moments of $f_{X(t)}$. For Wiener-based option pricing algorithms (Monte Carlo; Li et al., 2010; Alexander and Venkatramanan, 2012), the calibration algorithm provided the optimal set of diffusion volatilities (see Tables 10 and 11), incorporating the risk-neutral density moments information. Even though the market risk-neutral density is generally extracted from the market prices, future scenarios can be generated and priced only with changes to the cumulants, then, MGEE could be used for market risk sensitivity analysis.

There are four parameters of the density $g_{S(t)}$ that can be changed: $S_i(0), t, \rho_{i,j}$ and $\tilde{\sigma} = (\tilde{\sigma}_1, \ldots, \tilde{\sigma}_n)$ for $i \neq j \in \{1, \ldots, n\}$. Changes to $S_i(0)$ and to $t$ could appear more like a hedging exercise. Besides, changes to $\tilde{\sigma}$ are reflected over all the cumulants of the MVLN density. Then, $\tilde{\sigma}$ is selected as the parameter for the calibration. There are three objective functions $h_i(\tilde{\sigma})$: we will minimise each function for a different calibration:

$$h_2(\tilde{\sigma}) = \| M_{1,1,2} \|_2,$$
$$h_3(\tilde{\sigma}) = \| M_{1,1,2} \|_2 + \| M_{1,1,2,1} \|_2,$$
$$h_4(\tilde{\sigma}) = \| M_{1,1,2} \|_2 + \| M_{1,1,2,1} \|_2 + \| M_{1,1,2,1,1} \|_2.$$

Denote $\tilde{\sigma}$ to be the optimal volatility. Increments on $\tilde{\sigma}$ result in increments of the moments and the cumulants of $g_{S(t)}$. If the moments of $f_{X(t)}$ are lower than the moments of $g_{S(t)}$, the algorithm will decrease $\tilde{\sigma}$. The norm used is $\| \cdot \|_2$; However, other norms were tested with slower convergence rates towards the optimal value. The density $f_{X(t)}$ to be tested is the risk-neutral density of the multi-asset jump-diffusion process defined in Section 5, and it will be calibrated against different $\lambda, \nu, \sigma, r$, and $t$ parameters. The correlation between assets $\rho_{i,j}$ is set to zero. Since the multi-asset jump-diffusion process of Section 5 converges into a MVLN distribution, the optimal volatility value is:

$$\hat{\sigma}_i = \sigma_i + \lambda \nu_i.$$

If the optimal value is reached by the optimisation algorithm, with a low tolerance ($10^{-15}$) between the optimal parameter and the proposed solution, the objective function $h_i(\tilde{\sigma}), i \in \{2, 3, 4\}$ will be zero. For this reason, a noise effect is added to the algorithm, estimating the moments of $f_{X(t)}$ with the sample cumulants of a Monte Carlo simulation. Additionally, a maximum number of function evaluations is established for the optimisation.

For each case, the MGEE of zero- ($MGEE0$), second- ($MGEE2$), third- ($MGEE3$) and fourth- ($MGEE4$) order moments are calculated. The expansion of order $n$ includes the polynomials of order $n - 1, n - 2, \ldots, 1$. As the first-order cumulants are equal for any density, the first moment expansion $MGEE(1)$ is always equal to $MGEE0$; this is due to the arbitrage principle. The algorithm used to minimise $h_i(\sigma), i \in \{1, \ldots, 3\}$ is a
constrained convex optimisation method denoted as sequential quadratic programming (SQP). The implementation is in MATLAB with the *fmincon* function. A constraint over the volatility is that the covariance matrix $\Sigma$ must be positive semi-definite.

In the case $p_{1,j} = 0$, the minimisation of the objective function $h_2(\tilde{\sigma})$:

$$h_2(\tilde{\sigma}) = \|M(t)\|_2 = (k_{i,i}(f_{X(t)}) - k_{i,i}(g_{S(t)}))^2 + \cdots + (k_{n,n}(f_{X(t)}) - k_{n,n}(g_{S(t)}))^2,$$

has a closed-form solution, where $k_{i,i}(g_{S(t)})$ is the second central cumulant of $g_{S(t)}$, and $k_{i,i}(f_{X(t)})$ is the second cumulant of $f_{X(t)}$, a parameter given by the initial conditions of the problem. Equate the second-order cumulants of $f_{X(t)}$ and $g_{S(t)}$:

$$k_{i,i}(f_{X(t)}) = k_{i,i}(g_{S(t)}) = m_{i,i}(g_{S(t)}) - m_i(g_{S(t)})^2 = \exp\left(2\log(S_i(0)) + 2\tilde{\sigma}_i t + 2\left(r - \frac{1}{2}\sigma^2\right)t\right) - \exp\left(\log(S_i(0)) + rt\right)^2,$$

where $m_i(g_{S(t)})$ and $m_{i,i}(g_{S(t)})$ are the first- and second-order moments of $g_{S(t)}$. Clear the $\tilde{\sigma}_i$ variable, and the optimal solution yields:

$$\hat{\sigma}_i = \sqrt{\log(s_i(0))^2 + k_{i,i}(f_{X(t)})\exp(-2rt) - 2\log(s_i)}.$$

This solution is independent of the distribution of $f_{X(t)}$, and could be used when the correlations between the assets are zero.

### 6.2 Results

An inspection of the results demonstrates the effectiveness of the calibration method on precision. Table 4 shows the mean dollar error of the MGEE approximation, in a cross-pairs *objective function used for calibration – higher-order cumulant considered for MGEE approximation*. This is a resumé of Tables 12, 13 and 14, with the MGEE approximation over different jump-diffusion processes. There is an evident reduction of the mean dollar error when a calibration method is applied. The two best objective functions are $h_2(\tilde{\sigma})$ and $h_3(\tilde{\sigma})$. The optimal order of the cumulants to be considered for the expansion is the second (MGEE2). In some cases there exists an improvement in precision when the third-order cumulant is included in the expansion. The inclusion of the fourth-order cumulant reports highly noisy results. This is a consequence of the small cross-moments of the density $f_{X(t)}$ that exacerbate fourth-order cumulant differences.

The *Li2010* and *AV2012* methodologies underperformed the MGEE, although this time they incorporated the information of skewness and kurtosis with the calibration of the volatility. For the *Uncalibrated* MGEE, the MC column represents option prices over the multi-asset GBM process calculated with the Monte Carlo algorithm. For the calibrated MGEE, the $MC$, *Li2010* and *AV2012* columns represent the option prices after the GBM process were adjusted by the calibration method. Tables 7 and 8 show the improvement in the MGEE when moments of higher-order are added to the expansion; they were calculated subtracting columns MGEE2, MGEE3 and MGEE4 from the MC column in Tables 4 and 5. It is evident an improvement in the use of a second-order MGEE for extreme cases (Table 4 with $\lambda \in \{1,10\}$), and for less extreme cases (Table 5 with $\lambda = 1$) adding moments of second-, third- and fourth-order were beneficial to explain the price of the jumps in the risk-neutral density.

For Table 12 of the calibration of $h_2(\tilde{\sigma})$, column $MC$/MGEE2 contains the results of the option price over a multivariate GBM process with volatility adjusted by (20). These are the results of the best approximating MVLN distribution, and they will be the benchmark. In 56.25% (27 of 48 of the cases) the MGEE approximation price that included the second-order cumulant correction was superior. In 22.91% (11 of 48) of the cases the inclusion of the third-order cumulant, when $h_2(\tilde{\sigma})$ is used for calibration, will produce a better approximation. Table 6 shows a resumé of the number of best approximations for each pair *objective function used for calibration – higher-order cumulant considered for MGEE approximation*. The MGEE with the second-order expansion is the best approximation in 42.18% (81 of 192) of the total. The algorithms of *AV2012* and *Li2010* jointly, are just better in 7.81% (9 of 192) of the total cases. In Tables 10 and 11 the values of the optimal $\tilde{\sigma}_i$ parameters

---

14 Table 9 of results without calibration of the $\tilde{\sigma}_i$ parameter was considered for testing the effectiveness of the calibration algorithm.
are reported. The first column $\hat{\sigma}_i$ represents the optimal volatility given the jump-diffusion price process has a MVLN distribution. The initial volatility for all assets is $\sigma_i = 0.2$. The optimisation algorithm achieves results close to the optimal $\sigma$. For $\lambda = 10, \nu = 0.2$ the calibration algorithm outweighs the volatility. The small fourth-order cross-moments of the simulation increase the total kurtosis and the methodology the algorithm uses to reduce this is to increase the $\hat{\sigma}_i$ parameters beyond the optimal theoretical value.

Analysing the precision in Table 9, the third-order expansion ($MGEE^3$) is the best approximation in the majority of the cases (39.58%), while the second-order expansion ($MGEE^2$) achieves the best approximation in 27.08% of the cases. The increase in values of the parameters $\lambda, \nu, t$, and $r$ are reflected as an increase in the cumulants of the risk-neutral density $f_X(t)$. For example, for the parameters $\lambda = 10, \nu = 0.2$, the third- and fourth-order expansions, $MGEE^3$ and $MGEE^4$, only add noise to the approximation. However, these values are extreme for the real market data values reported by Das and Uppal (2004). In Table 5 there is the mean dollar error when only processes with $\lambda = 1$ are considered. The improvement in the fourth-order approximation is significant. Tables 13 and 14 provide similar results. They are ineffective only when the parameters of the jump-diffusion $\lambda, \nu$ are extreme. The aggregation of higher moments in the calibrating function $h_i(\hat{\sigma}_i)$, produce a ripple effect over the precision of the immediate previous-order MGEE approximation $MGEE(i - 1)$, when compared with the $MGEE(i)$, when the calibration is done with $h_i(\hat{\sigma}_i)$.

Table 15 displays the percentage of simulated paths that result in a positive value when they are evaluated over the generated MGEE density. The improvement given by the calibration is significant. The best pair is the second column with ($h_3(\hat{\sigma}_i), MGEE^2$). This pair is a MVLN with the volatility calculated by (20), and it is the benchmark method. The results on pair ($h_3(\hat{\sigma}_i), MGEE^3$) reveal results close to the density function. For $\lambda = 10, \nu = 0.20$ there is a increase of the negative region; this could be used to measure the performance of the results in the case where the market option price is unknown.

### Table 4: Mean dollar error of MGEE approximations of jump-diffusion processes. The columns $MGEE^2$, $MGEE^3$, and $MGEE^4$ of the rows Uncalibrated, $h_2(\hat{\sigma}_i)$, $h_3(\hat{\sigma}_i)$, $h_4(\hat{\sigma}_i)$ are the error-averages of the 48 cases of Tables 9, 12, 13, and 14, respectively. The column ‘Wiener-MC’ represents the average difference between the uncalibrated ‘Wiener’ process and the calibrated ‘Wiener’ process (all cases including $\lambda = 1$ and $\lambda = 10$).

<table>
<thead>
<tr>
<th>Objective Function</th>
<th>Wiener-MC</th>
<th>AV2012</th>
<th>Li2012</th>
<th>MGEE2</th>
<th>MGEE3</th>
<th>MGEE4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uncalibrated</td>
<td>0.2441</td>
<td>0.6028</td>
<td>0.3105</td>
<td>0.1531*</td>
<td>0.3312</td>
<td>2.4641</td>
</tr>
<tr>
<td>$h_2(\hat{\sigma}_i)$</td>
<td>-</td>
<td>0.8468</td>
<td>0.2283</td>
<td>0.0122*</td>
<td>0.0870</td>
<td>32.5970</td>
</tr>
<tr>
<td>$h_3(\hat{\sigma}_i)$</td>
<td>0.0783</td>
<td>1.1194</td>
<td>0.3736</td>
<td>0.0115*</td>
<td>0.1371</td>
<td>18.4856</td>
</tr>
<tr>
<td>$h_4(\hat{\sigma}_i)$</td>
<td>0.1838</td>
<td>1.3484</td>
<td>0.4786</td>
<td>0.0353*</td>
<td>0.1856</td>
<td>11.1462</td>
</tr>
</tbody>
</table>

### Table 5: Mean dollar error of MGEE approximations of jump-diffusion processes. The columns $MGEE^2$, $MGEE^3$ and $MGEE^4$ of the rows Uncalibrated, $h_2(\hat{\sigma}_i)$, $h_3(\hat{\sigma}_i)$, $h_4(\hat{\sigma}_i)$ are the error-averages of the 48 cases of Tables 9, 12, 13 and 14, respectively. The column ‘Wiener-MC’ represents the average difference between the uncalibrated ‘Wiener’ process and the calibrated ‘Wiener’ process (only cases with $\lambda = 1$).

<table>
<thead>
<tr>
<th>Objective Function</th>
<th>Wiener-MC</th>
<th>AV2012</th>
<th>Li2012</th>
<th>MGEE2</th>
<th>MGEE3</th>
<th>MGEE4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uncalibrated</td>
<td>0.1402</td>
<td>0.7479</td>
<td>0.2156</td>
<td>0.0506</td>
<td>0.0426*</td>
<td>0.1374</td>
</tr>
<tr>
<td>$h_2(\hat{\sigma}_i)$</td>
<td>-</td>
<td>1.1364</td>
<td>0.2073</td>
<td>0.0133*</td>
<td>0.0157</td>
<td>0.0303</td>
</tr>
<tr>
<td>$h_3(\hat{\sigma}_i)$</td>
<td>0.1028</td>
<td>1.5575</td>
<td>0.4113</td>
<td>0.0103*</td>
<td>0.0159</td>
<td>0.0699</td>
</tr>
<tr>
<td>$h_4(\hat{\sigma}_i)$</td>
<td>0.2607</td>
<td>1.9107</td>
<td>0.5734</td>
<td>0.0418</td>
<td>0.0279*</td>
<td>0.1021</td>
</tr>
</tbody>
</table>
Table 6: Number of best approximations of jump-diffusion processes for different calibration methods. The columns MGEE2, MGEE3, and MGEE4 of the rows Uncalibrated, $h_2(\tilde{\sigma})$, $h_3(\tilde{\sigma})$, $h_4(\tilde{\sigma})$ are the number of cases with best approximation from the 48 different cases of Tables 9, 12, 13, and 14, respectively. The column ‘Wiener-MC’ represents the case where there is no improvement with a MGEE of second-, third- or fourth-order.

<table>
<thead>
<tr>
<th>Objective Function</th>
<th>Wiener-MC</th>
<th>AV2012</th>
<th>Li2010</th>
<th>MGEE2</th>
<th>MGEE3</th>
<th>MGEE4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uncalibrated</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>13</td>
<td>19</td>
<td>10</td>
</tr>
<tr>
<td>$h_2(\tilde{\sigma})$</td>
<td>-</td>
<td>0</td>
<td>3</td>
<td>27*</td>
<td>13</td>
<td>5</td>
</tr>
<tr>
<td>$h_3(\tilde{\sigma})$</td>
<td>4</td>
<td>0</td>
<td>3</td>
<td>24*</td>
<td>11</td>
<td>6</td>
</tr>
<tr>
<td>$h_4(\tilde{\sigma})$</td>
<td>3</td>
<td>0</td>
<td>3</td>
<td>17</td>
<td>19*</td>
<td>6</td>
</tr>
<tr>
<td>Total</td>
<td>7</td>
<td>6</td>
<td>9</td>
<td>81</td>
<td>62</td>
<td>27</td>
</tr>
</tbody>
</table>

Table 7: Percentage improvement when moments of higher-order are included in the MGEE approximations of jump-diffusion processes for different calibration methods ($\lambda = 1$ and $\lambda = 10$). The columns MGEE2, MGEE3, and MGEE4 of the rows Uncalibrated, $h_2(\tilde{\sigma})$, $h_3(\tilde{\sigma})$, $h_4(\tilde{\sigma})$ are the percentage improvement in the error-averages of the 48 cases of Tables 9, 12, 13, and 14, respectively (all cases including $\lambda = 1$ and $\lambda = 10$).

<table>
<thead>
<tr>
<th>Objective Function</th>
<th>MGEE2</th>
<th>MGEE3</th>
<th>MGEE4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uncalibrated</td>
<td>0.0910*</td>
<td>-0.0871</td>
<td>-2.2200</td>
</tr>
<tr>
<td>$h_2(\tilde{\sigma}) = |M_{t_1,t_2}|_2$</td>
<td>-0.0122*</td>
<td>-0.0870</td>
<td>-32.5970</td>
</tr>
<tr>
<td>$h_3(\tilde{\sigma}) = |M_{t_1,t_2}|<em>2 + |M</em>{t_1,t_2,t_3}|_2$</td>
<td>0.0668*</td>
<td>-0.0588</td>
<td>-18.4073</td>
</tr>
<tr>
<td>$h_4(\tilde{\sigma}) = |M_{t_1,t_2}|<em>2 + |M</em>{t_1,t_2,t_3}|<em>2 + |M</em>{t_1,t_2,t_3,t_4}|_2$</td>
<td>0.1485*</td>
<td>-0.0018</td>
<td>-10.9624</td>
</tr>
</tbody>
</table>

Table 8: Percentage improvement when moments of higher-order are included in the MGEE approximations of jump-diffusion processes for different calibration methods (only cases with $\lambda = 1$). The columns MGEE2, MGEE3, and MGEE4 of the rows Uncalibrated, $h_2(\tilde{\sigma})$, $h_3(\tilde{\sigma})$, $h_4(\tilde{\sigma})$ are the percentage improvement in the error-averages of the 48 cases of Tables 9, 12, 13, and 14, respectively (only cases with $\lambda = 1$).

<table>
<thead>
<tr>
<th>Objective Function</th>
<th>MGEE2</th>
<th>MGEE3</th>
<th>MGEE4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Uncalibrated</td>
<td>0.0896</td>
<td>0.0976*</td>
<td>0.0028</td>
</tr>
<tr>
<td>$h_2(\tilde{\sigma}) = |M_{t_1,t_2}|_2$</td>
<td>-0.0133*</td>
<td>-0.0157</td>
<td>-0.0303</td>
</tr>
<tr>
<td>$h_3(\tilde{\sigma}) = |M_{t_1,t_2}|<em>2 + |M</em>{t_1,t_2,t_3}|_2$</td>
<td>0.0925*</td>
<td>0.0869</td>
<td>0.0329</td>
</tr>
<tr>
<td>$h_4(\tilde{\sigma}) = |M_{t_1,t_2}|<em>2 + |M</em>{t_1,t_2,t_3}|<em>2 + |M</em>{t_1,t_2,t_3,t_4}|_2$</td>
<td>0.2189</td>
<td>0.2328*</td>
<td>0.1586</td>
</tr>
</tbody>
</table>

7 CONCLUSIONS

The theory of multi-asset option pricing has been developed under the concept of approximating the multivariate risk-neutral density of the assets with the univariate density of the payoff function, undermining important information contained in the dependence structure of the multivariate density.\(^\text{15}\) There exist several approximations for multi-asset option pricing;\(^\text{16}\) however, there is no pricing formula at the time of writing this research that accounts for the effects of the higher-order cross-moments.

In this research, an approximation termed the Multivariate Generalised Edgeworth Expansion (MGEE) is used to fit the unknown risk-neutral density with a known auxiliary continuous density through the difference in the moments of the risk-neutral density. The expansion can enhance a distribution fit for options over densities with high skewness or high kurtosis. The method is intended for European options. Nevertheless, if a risk-neutral density from a path-dependent option can be estimated, then the methodology can be applied. When the multivariate lognormal (MVLN) distribution is used as the auxiliary distribution, the option pricing formula reveals three components: a Wiener component, the moments corrections, and the error term caused by including only the first four coefficients in the approximation. The MGEE approximation is related to variance swaps, and a new contract defined as moment option is proposed as a future extension of our work. Likewise,

\(^{15}\)As motivation, in Section 2.3, a hedging example of the tracking error of discrete hedging with the univariate density of the sum of lognormals is presented.

\(^{16}\)See Kristensen and Mele (2011); Li et al. (2010) and Alexander and Venkatramanan (2012).
the partial derivatives of the expansion are sensitivities of the risk-neutral density against changes in its form, translation, dispersion, skewness, and heavy-tailedness, results that are important for future topics of research.

A multivariate Merton’s jump-diffusion process is used to test the MGEE option pricing approximation. There are two initial purposes for the approximation: the first is not obvious at first sight, and it is to find the price premium that accounts for the difference in the moments. Risk managers could check whether the difference in the price of two risk-neutral densities come from the second-order moments (greater volatility of the risk-neutral density), third-order moments (positive/negative bias, and bias of the density over an asset), or fourth-order moments (heavy-tailedness of the distribution). Several examples with dimension $n = 2$ and $n = 5$ are presented in Sections 4.7 and 5.2, where the relationship between an increase in certain moments is related to the increase in price. The analytical multi-asset pricing methodologies of Li et al. (2010) and Alexander and Venkatramanan (2012) were used for comparison purposes. The results show that when the information provided by the moments of the risk-neutral density is provided, the MGEE improves the results of a Monte Carlo methodology, and is superior to the Li et al. (2010) and Alexander and Venkatramanan (2012) analytical approximations.

The second purpose of the MGEE is the price approximation: a calibration over the MVLN auxiliary density parameters to reduce the moments’ differences with the risk-neutral density is proposed and developed in Section 6. The results of pricing the options over jump-diffusion processes show that the mean dollar error for the approximation could be of $\sim 1\% - 1.5\%$ when only second-order moments are included in the expansion. These results include the extreme case of jump-diffusions with $\lambda = 10$. When the set examples are filtered to the case of $\lambda = 1$, the mean dollar error when third- and fourth-order moments are included in the expansion improves substantially. This is still a high parameter value for equity market returns. A major drawback of the MGEE expansion option pricing is the running performance of the algorithm, which is slow compared with the Monte Carlo algorithm (8 times slower when including the fourth-order moments in the expansion).

The results of this research are only the initial step to further investigations. Important extensions to our work include the development of a new theory for hedging the cross-moments of the risk-neutral density of multi-asset contract. The instruments to achieve this goal are defined as the moment options, and they require an extensive study and development. In a similar way, extensions to the theory of the sensitivity of the risk-neutral distribution to changes in the moments is proposed as a considerable area of investigation.

Risk managers and hedgers will benefit from having an option formula that accounts for the moments of the risk-neutral distributions. Although the results are for general multi-assets contracts, the whole set of univariate option contracts could enhance their performance if the theory of moments developed in this work were applied to their pricing and hedging.

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17 Das and Uppal (2004) estimate a $\lambda < 0.1$ for emerging equity markets.
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Schlögl, E. (2013), ‘Option pricing where the underlying assets follow a gram/charlier density of arbitrary order’, *Journal of Economic Dynamics & Control* 37, 611–632.


A.1 Proof of Proposition 3.1

Proof. Let \( f_X \) be the continuous-time function density of \( X \) and \( \xi \in \mathbb{R}^N \). The characteristic function (CF) of \( X \) is defined as:

\[
\psi(x, \xi) = \mathbb{E} \left[ \exp \left( \xi_t X_t \right) \right].
\]

Denote \( m_{t_1, \ldots, t_p}(x) \) the p-moment of \( X \). This function can be expanded into the infinite series:

\[
\psi(x, \xi) = 1 + \sum_{j=1}^{n-1} \xi_{t_j} m_{t_1, \ldots, t_j}(x) i^j / j! + o(\|\xi\|^n), \tag{21}
\]

which is convergent for small \( \xi \). We calculate the log function:

\[
\log \psi(x, \xi) = \sum_{j=1}^{n-1} \xi_{t_j} \frac{m_{t_1, \ldots, t_j}(x) i^j}{j!} + o(\|\xi\|^n),
\]

where \( k_{t_1, \ldots, t_p}(s) \) are the cumulants of order \( p \) of \( X \). Now suppose that we have another continuous density function \( g_s \) of a stochastic process \( s \). Then we can write:

\[
\log \psi(x, \xi) = \sum_{j=1}^{n-1} \xi_{t_j} \frac{k_{t_1, \ldots, t_j}(s) i^j}{j!} + o(\|\xi\|^n),
\]

where \( k_{t_1, \ldots, t_p}(s) \) are the cumulants of order \( p \) of \( s \). Applying the exponential function on both sides:

\[
\psi(x, \xi) = \exp \left( \sum_{j=1}^{n-1} \xi_{t_j} \frac{k_{t_1, \ldots, t_j}(s) i^j}{j!} \right) \psi(s, \xi) + o(\|\xi\|^n).
\]

It could be demonstrated that: \( \exp(o(\|\xi\|^n)) = 1 + o(\|\xi\|^n) \). But the exponential function can be expanded as in (21). Then,

\[
\psi(x, \xi) = \left( \sum_{j=1}^{n-1} \xi_{t_j} M_{t_1, \ldots, t_j} \frac{i^j}{j!} \right) \psi(s, \xi) + o(\|\xi\|^n), \tag{22}
\]

where \( M_{t_1, \ldots, t_j} \) are the difference of the moments of distributions \( f_X, g_s \).

The Fourier transforms of \( f_X \) and \( g_s \) are, respectively:

\[
f_X = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\xi' x) \psi(x, \xi) d\xi, \tag{23}
\]

\[
g_s = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\xi' s) \psi(s, \xi) d\xi, \tag{24}
\]

and the \( j \)-partial derivative of (24) is:

\[
(-1)^j \frac{\partial^j}{\partial s_{t_1} \ldots \partial s_{t_j}} g_s = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\xi' s) i^j \xi_{t_1} \ldots \xi_{t_j} \psi(s, \xi) d\xi. \tag{25}
\]
Applying the inverse Fourier transform to (22), we have that,
\[
\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\xi' x) \psi(x, \xi) d\xi = \\
\frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(-i\xi' s) \left( \sum_{j=0}^{n-1} \xi_{[i_1]} \cdots \xi_{[i_j]} M_{[i_1, \ldots, [j]]} \psi(s, \xi) + o(\|\xi\|^n) \right) d\xi.
\]
Using (23), (24), and (25), it finally yields,
\[
f_x = gs + \sum_{j=1}^{n-1} M_{[i_1, \ldots, [j]]} \left( \frac{(-1)^j}{j!} \frac{\partial^j}{\partial s_{i_1} \cdots \partial s_{i_j}} gs + \varepsilon(s, n) \right),
\]
where
\[
\varepsilon(s, n) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(i\xi' s) o(\|\xi\|^n) d\xi.
\]

A.2 First four partial derivatives of a MVLN distribution

Denote \(\Sigma^{-1}_n\) the inverse matrix of \(\Sigma_n\) as in (19) and define \(\Lambda = \frac{1}{2} (\log(S) - \mu_s)' \Sigma^{-1}_n (\log(S) - \mu_s)\). The first four terms of \(\frac{\partial^2 \pi_n}{\partial s_{i_1} \cdots \partial s_{i_3}} gs(s)\) are:

The first-order partial derivative is,
\[
\frac{\partial}{\partial S_{i_1}} gs = gs \left( -\frac{1}{S_{i_1}} + \frac{\partial \Lambda}{\partial S_{i_1}} \right), \tag{26}
\]
where,
\[
\frac{\partial \Lambda}{\partial S_{i_1}} = -\frac{1}{S_{i_1}} \Sigma^{-1}_{n, (i_1, :)} (\log(S) - \mu_s)
\]
and \(\Sigma^{-1}_{n, (i_1, :)}\) is the \(i_1\)-th row of \(\Sigma^{-1}_n\). The second-order partial derivatives of \(gs\) are,
\[
\frac{\partial^2}{\partial S_{i_1}^2} gs = gs \left( \frac{2}{S_{i_1}^2} \frac{\partial \Lambda}{\partial S_{i_1}} + \left( \frac{\partial \Lambda}{\partial S_{i_1}} \right)^2 + \frac{\partial^2 \Lambda}{\partial S_{i_1}^2} \right),
\]
\[
\frac{\partial^2}{\partial S_{i_1} \partial S_{i_2}} gs = gs \left( \frac{1}{S_{i_1} S_{i_2}} \frac{\partial \Lambda}{\partial S_{i_1}} - \frac{1}{S_{i_1} S_{i_2}} \frac{\partial \Lambda}{\partial S_{i_2}} + \frac{1}{S_{i_1} S_{i_2}} \frac{\partial \Lambda}{\partial S_{i_1}} + \frac{\partial \Lambda}{\partial S_{i_2}} + \frac{\partial^2 \Lambda}{\partial S_{i_1} \partial S_{i_2}} \right), \tag{27}
\]
where,
\[
\frac{\partial^2 \Lambda}{\partial S_{i_1}^2} = \frac{1}{S_{i_1}^2} \left( \Sigma^{-1}_{n, (i_1, :)} (\log(S) - \mu_s) - \xi_{1, i_1} \right),
\]
\[
\frac{\partial^2 \Lambda}{\partial S_{i_1} \partial S_{i_2}} = -\frac{1}{S_{i_1} S_{i_2}} \xi_{1, i_2}.
\]

The third-order partial derivatives of \(gs\) are,
\[
\frac{\partial^3}{\partial S_{i_1}^3} gs = gs \left( \frac{6}{S_{i_1}^3} \frac{\partial \Lambda}{\partial S_{i_1}} - \frac{3}{S_{i_1}^2} \frac{\partial \Lambda}{\partial S_{i_1}} \left( \frac{\partial \Lambda}{\partial S_{i_1}} \right)^2 - \frac{3}{S_{i_1}^2} \frac{\partial \Lambda}{\partial S_{i_1}} \frac{\partial \Lambda}{\partial S_{i_1}} + \frac{3}{S_{i_1} S_{i_2}} \frac{\partial \Lambda}{\partial S_{i_1}} + \frac{\partial \Lambda}{\partial S_{i_1}} \frac{\partial \Lambda}{\partial S_{i_2}} + \frac{\partial \Lambda}{\partial S_{i_1}} \frac{\partial \Lambda}{\partial S_{i_2}} + \frac{\partial \Lambda}{\partial S_{i_1}} \right),
\]
\[
\frac{\partial^3}{\partial S_{i_1}^2 \partial S_{i_2}} gs = \left( \frac{1}{S_{i_1}^2} + \frac{\partial \Lambda}{\partial S_{i_1}} \right) \frac{\partial^2}{\partial S_{i_1}^2} gs - gs \left( \frac{2}{S_{i_1} S_{i_1} S_{i_2}} \frac{\partial \Lambda}{\partial S_{i_1}} + \frac{2}{S_{i_1} S_{i_1} S_{i_2}} \frac{\partial \Lambda}{\partial S_{i_1}} + \frac{\partial \Lambda}{\partial S_{i_1}} \right),
\]
\[
\frac{\partial^3}{\partial S_{i_1} \partial S_{i_2} \partial S_{i_3}} gs = \left( \frac{1}{S_{i_1} S_{i_2} S_{i_3}} \frac{\partial^2}{\partial S_{i_1} \partial S_{i_2}} gs + gs \left( \frac{1}{S_{i_1} S_{i_2} S_{i_3} S_{i_2}} \frac{\partial \Lambda}{\partial S_{i_1}} - \frac{1}{S_{i_2} S_{i_1} S_{i_2} \partial S_{i_1}} + \frac{\partial \Lambda}{\partial S_{i_1}} \frac{\partial \Lambda}{\partial S_{i_1}} + \frac{\partial \Lambda}{\partial S_{i_2}} \frac{\partial \Lambda}{\partial S_{i_2}} \right) \right), \tag{28}
\]
where,
\[
\frac{\partial^3 \Lambda}{\partial S_1^3} = \frac{1}{S_1^3} \left( -2S_1^{-1} (\log (S) - \mu_s) + 3s_{1,t_1} \right),
\]
\[
\frac{\partial^3 \Lambda}{\partial S_1^2 \partial S_2} = \frac{1}{S_1^2 S_2} g_{s_{1,t_2}}.
\]

The term \( \frac{\partial^3 \Lambda}{\partial S_1 \partial S_2 \partial S_3} \) is equal to zero.

And the fourth-order partial derivatives of \( g_s \) are,
\[
\frac{\partial^4 g_s}{\partial S_1^4} = g_s \left( \frac{24}{S_1^4} - \frac{24}{S_1^2 S_2} \frac{\partial \Lambda}{\partial S_1} + \frac{12}{S_1^2} \left( \frac{\partial^2 \Lambda}{\partial S_1^2} \right) + \frac{12}{S_1^2} \left( \frac{\partial \Lambda}{\partial S_1} \right)^2 - \frac{12}{S_1^2} \frac{\partial \Lambda}{\partial S_1} \frac{\partial^2 \Lambda}{\partial S_1^2} - \frac{4}{S_1^2} \frac{\partial^3 \Lambda}{\partial S_1^3} \right),
\]
\[
\frac{\partial^4 g_s}{\partial S_1^3 \partial S_2} = \left( \frac{-1}{S_1^3} + \frac{\partial \Lambda}{\partial S_1} \right) \frac{\partial^3 g_s}{\partial S_1^2 \partial S_2} + g_s \left( \frac{2}{S_1^2 S_2} - \frac{2}{S_1^2} \frac{\partial \Lambda}{\partial S_1} + \frac{2}{S_1^2} \frac{\partial^2 \Lambda}{\partial S_1^2} \right)
\]
\[
\frac{\partial^4 g_s}{\partial S_1^2 \partial S_2 \partial S_3} = \left( \frac{-1}{S_1^2 S_2} + \frac{\partial \Lambda}{\partial S_1} \right) \frac{\partial^3 g_s}{\partial S_1 \partial S_2^2} + g_s \left( \frac{2}{S_1^2 S_2} - \frac{2}{S_1^2} \frac{\partial \Lambda}{\partial S_1} + \frac{2}{S_1^2} \frac{\partial^2 \Lambda}{\partial S_1^2} \right)
\]
\[
\frac{\partial^4 g_s}{\partial S_1 \partial S_2 \partial S_3^2} = \left( \frac{-1}{S_1^2 S_2} + \frac{\partial \Lambda}{\partial S_1} \right) \frac{\partial^3 g_s}{\partial S_1 \partial S_2 \partial S_3} + g_s \left( \frac{2}{S_1^2 S_2} - \frac{2}{S_1^2} \frac{\partial \Lambda}{\partial S_1} + \frac{2}{S_1^2} \frac{\partial^2 \Lambda}{\partial S_1^2} \right)
\]
\[
\frac{\partial^4 g_s}{\partial S_2^4} = \left( \frac{-1}{S_2^4} + \frac{\partial \Lambda}{\partial S_2} \right) \frac{\partial^3 g_s}{\partial S_2^3} + g_s \left( \frac{2}{S_1 S_2^3} - \frac{2}{S_2} \frac{\partial \Lambda}{\partial S_2} + \frac{2}{S_2} \frac{\partial^2 \Lambda}{\partial S_2^2} \right)
\]

(29)
where,
\[
\frac{\partial^4 \Lambda}{\partial S_{l_1}^4} = \frac{1}{S_{l_1}^4} \left( 6\Sigma_{s,(l_1:)}^{-1} (\log(\mathbf{S}) - \mu_s) - 11\zeta_{l_1,l_1} \right),
\]
\[
\frac{\partial^4 \Lambda}{\partial S_{l_1}^3 \partial S_{l_2}} = -\frac{2}{S_{l_1}^4 S_{l_2}} \zeta_{l_1,l_2},
\]
\[
\frac{\partial^4 \Lambda}{\partial S_{l_1}^2 \partial S_{l_2}^2} = -\frac{1}{S_{l_1}^2 S_{l_2}^2} \zeta_{l_1,l_2}.
\]
The terms \(\frac{\partial^4 \Lambda}{\partial S_{l_1}^4 \partial S_{l_2}}, \frac{\partial^4 \Lambda}{\partial S_{l_1}^3 \partial S_{l_2} \partial S_{l_3}}, \text{ and } \frac{\partial^4 \Lambda}{\partial S_{l_1}^2 \partial S_{l_2} \partial S_{l_3} \partial S_{l_4}}\) are equal to zero.

### A.3 Analysis of the correction term \(C_{0,W_0[1,...,J]}(\Pi(\mathbf{x}(t)))\)

The first-order partial derivative of \(g_s\) is,
\[
\frac{\partial}{\partial S_{l_1}} g_s = g_s \left( -\frac{1}{S_{l_1}} + \frac{\partial \Lambda}{\partial S_{l_1}} \right) = g_s \left( -\frac{1}{S_{l_1}} \right) \left( 1 + \Sigma_{s,(l_1:)}^{-1} \log(\mathbf{S}) - \Sigma_{s,(l_1:)}^{-1} \mu_s \right),
\]
where \(\Sigma_{s,(l_1:)}^{-1}\) is the \(l_1\)-th row of \(\Sigma_s^{-1}\).

Disentangling the partial derivative we get:

1. The terms \(g_s \left( -\frac{1}{S_{l_1}} \right)\) and \(g_s \left( -\frac{1}{S_{l_1}} \Sigma_{s,(l_1:)}^{-1} \mu_s \right)\) could be re-expressed as MVLN densities:

**Proposition A.1.** Let \(\mathbf{S}(t) = (S_1(t), \ldots, S_n(t))\) be a multivariate GBM process defined as in (1), with MVLN distribution and parameters \(\mu_s, \Sigma_s\), the conditional expected value:

\[
\mathbb{E}_0^Q \left[ \log(S_1(t))^{\alpha_1} \ldots \log(S_n(t))^{\alpha_n} S_1(t)^{\beta_1} \ldots S_n(t)^{\beta_n} | S_j(t) \geq K \right],
\]

with \(j \in \{1, \ldots, n\}\), \(\alpha_i, \beta_i \in \mathbb{R}\), is equal to the lower truncated moment of a MVN process \(\mathbf{Y}(t) = (Y_1(t), \ldots, Y_n(t))\) times a constant \(A\):

\[
A \cdot \mathbb{E}_0^Q \left[ Y_1(t)^{\alpha_1} \ldots Y_n(t)^{\alpha_n} | Y_j(t) \geq \log(K) \right],
\]

where \(\mathbf{Y}(t) \sim N(\mu_s + \Sigma_s \beta, \Sigma_s)\).

**Proof.** Applying the definition of the MVLN, and after the change of variable \(S_i = \exp(Y_i)\), we have the resulting moment over the MVN distribution:

\[
\mathbb{E}_0^Q \left[ \log(S_1(t))^{\alpha_1} \ldots \log(S_n(t))^{\alpha_n} S_1(t)^{\beta_1} \ldots S_n(t)^{\beta_n} | S_j(t) \geq K \right] = L_1^{-1} \left( \int_0^{\infty} \int_K^\infty (2\pi)^{-n/2} |\Sigma_s|^{-1/2} \left( \prod_{i=1}^{n} \log(s_i)^{\alpha_i} \right) \left( \prod_{i=1}^{n} s_i^{\beta_i-1} \right) \exp \left( -\frac{1}{2} (\log(s) - \mu_s)^T \Sigma_s^{-1} (\log(s) - \mu_s) \right) ds \right) \]

\[
= L_1^{-1} \left( \int_{\log(K)}^{\infty} (2\pi)^{-n/2} |\Sigma_s|^{-1/2} \left( \prod_{i=1}^{n} y_i^{\alpha_i} \right) \exp \left( y^\top \beta \right) \exp \left( -\frac{1}{2} (y - \mu_s)^T \Sigma_s^{-1} (y - \mu_s) \right) dy \right),
\]

where \(L_1 = \mathbb{P}^Q(S_j(t) \geq K)\). The time parameter of the process \(\mathbf{Y}(t)\) is omitted to simplify the notation. The last expression can be transformed as:

\[
= L_1^{-1} \left( \int_{\log(K)}^{\infty} (2\pi)^{-n/2} |\Sigma_s|^{-1/2} \left( \prod_{i=1}^{n} y_i^{\alpha_i} \right) \exp \left( \frac{1}{2} \gamma \Sigma_s \beta + \beta^\top \mu_s - \frac{1}{2} (y - \zeta)^T \Sigma_s^{-1} (y - \zeta) \right) dy \right),
\]

with \(\zeta = \mu_s + \Sigma_s \beta\). Define \(L_2 = \mathbb{P}^Q(Y_j(t) \geq \log(K))\), then the last expression becomes:

\[
= \exp \left( \frac{1}{2} \gamma \Sigma_s \beta + \beta^\top \mu_s \right) \left( \frac{L_2}{L_1} \right) \mathbb{E}_0^Q \left[ Y_1(t)^{\alpha_1} \ldots Y_n(t)^{\alpha_n} | Y_j(t) \geq \log(K) \right].
\]
Then the variable \( Y(t) \) is distributed \( N(\mu_s + \Sigma_s\beta, \Sigma_s) \), the constant \( A = \exp \left( \frac{1}{2} \beta' \Sigma_s \beta + \beta' \mu_s \right) \left( \frac{S}{\sqrt{t}} \right) \) and the result follows.

**Corollary A.2.** Let \( gs \) have the multivariate density as in (17). Denote \( \alpha = (\alpha_1, \ldots, \alpha_n) \) a vector of integers. The function:

\[
g_s = S_1^{\alpha_1} \cdots S_n^{\alpha_n} gs,
\]

is a MVLN density function and can be re-written as:

\[
g_s = \exp(\mu_s^* \alpha)gs.
\]

**Proof.** With some algebraic calculations it follows from Proposition (A.1).

Then, the terms \( gs \left( -\frac{1}{S_{l_1}} \right) \) and \( gs \left( -\frac{1}{S_{l_1}} \right) \Sigma_{s,(l_1,:)}^{-1}\mu_s \) are MLVN densities, setting \( \alpha_{l_1} = -1, \alpha_i = 0, i \in \{1, \ldots, n\}, i \neq l_1 \) in Corollary (A.2), and we have the resulting density:

\[
g_s = -gs \left( \frac{1}{S_{l_1}} \right) = \exp(-\mu_1)gs,
\]

The term \( \Sigma_{s,(l_1,:)}^{-1}\mu_s \) is an scalar value, it is the dot product of two vectors.

1b. The term \( gs \left( -\frac{1}{S_{l_1}} \right) \Sigma_{s,(l_1,:)}^{-1}\log(s) \) is a sum of MLVN densities times a log-contract.

The first-order moment correction of (18) becomes:

\[
\exp(-rt) \sum_{l_1=1}^{n} M_{l_1}(-1) \int_0^{\infty} \Pi(s(t)) \frac{\partial}{\partial S_{l_1}} gs = \exp(-rt) \left( \sum_{l_1=1}^{n} M_{l_1}(-1)(\Sigma_{s,(l_1,:)}^{-1}\mu_s - 1) \exp(-\mu_1) \int_0^{\infty} \Pi(s(t))gs + \sum_{l_1=1}^{n} M_{l_1}(-1) \exp(-\mu_1) \sum_{j=1}^{n} \int_0^{\infty} \log(S_j)\Pi(s(t))gs \right).
\]

The second-order partial derivative of \( gs \) against the same variable is:

\[
\frac{\partial^2}{\partial S_{l_1}^2} gs = gs \left( \frac{2}{S_{l_1}} - 2\frac{\partial \Lambda}{\partial S_{l_1}} + \left( \frac{\partial \Lambda}{\partial S_{l_1}} \right)^2 + \frac{\partial^2 \Lambda}{\partial S_{l_1}^2} \right) = gs \left( \frac{1}{S_{l_1}^2} \right) \left( 2 + \left( \Sigma_{s,(l_1,:)}^{-1}\log(s) - \Sigma_{s,(l_1,:)}^{-1}\mu_s \right) + \left( \Sigma_{s,(l_1,:)}^{-1}\log(s) - \Sigma_{s,(l_1,:)}^{-1}\mu_s \right)^2 + \left( \Sigma_{s,(l_1,:)}^{-1}\log(s) - \Sigma_{s,(l_1,:)}^{-1}\mu_s - \varsigma_{l_1,l_1} \right) \right),
\]

\[
= gs \left( \frac{1}{S_{l_1}^2} \right) \left( 2 + \left( \Sigma_{s,(l_1,:)}^{-1}\mu_s + \left( \Sigma_{s,(l_1,:)}^{-1}\mu_s \right)^2 - \varsigma_{l_1,l_1} \right) + \Sigma_{s,(l_1,:)}^{-1}\log(s) \left( 3 - 2\Sigma_{s,(l_1,:)}^{-1}\mu_s \right) + \left( \Sigma_{s,(l_1,:)}^{-1}\log(s) \right)^2 \right).
\]

Unfolding we get three terms:

2a. The term \( gs \left( \frac{1}{S_{l_1}^3} \right) \left( 2 + \left( \Sigma_{s,(l_1,:)}^{-1}\mu_s + \left( \Sigma_{s,(l_1,:)}^{-1}\mu_s \right)^2 - \varsigma_{l_1,l_1} \right) \right) \) could be re-written as a MVLN density:

In this case, set \( \alpha_{l_1} = -2, \alpha_i = 0, i \in \{1, \ldots, n\}, i \neq l_1 \) in the Corollary (A.2), and the resulting density
yields:
\[
\hat{g}_s = gs \left( \frac{1}{S_{t_1}} \right) \left( 2 - 3\Sigma^{-1}_{s(t_1::)} \mu_s + \left( \Sigma^{-1}_{s(t_1::)} \mu_s \right)^2 = \xi_{t_1,t_1} \right) 
\]
\[
= \exp(-2\mu_t) \left( 2 - 3\Sigma^{-1}_{s(t_1::)} \mu_s + \left( \Sigma^{-1}_{s(t_1::)} \mu_s \right)^2 = \xi_{t_1,t_1} \right) g_s.
\]

2b. The term \( gs \left( \frac{1}{S_{t_1}} \right) \Sigma^{-1}_{s(t_1::)} \log(s) \left( 3 - 2\Sigma^{-1}_{s(t_1::)} \mu_s \right) \) is a sum of MVLN densities times a log-contract.

2c. The term \( gs \left( \frac{1}{S_{t_1}} \right) \left( \Sigma^{-1}_{s(t_1::)} \log(s) \right)^2 \) is a sum of MVLN densities times quadratic functions of log-contracts.

Resuming we have the second-order moment correction of (18) with the same index \( t_1 \) is:
\[
\exp(-rt) \sum_{l_1=1}^{n} M_{l_1,t_1} \frac{1}{2} \int_{0}^{\infty} \frac{\partial^2}{\partial S^2} g_s = 
\exp(-rt) \left( \sum_{l_1=1}^{n} M_{l_1,t_1} \frac{1}{2} \exp(-2\mu_t) \left( 2 - 3\Sigma^{-1}_{s(t_1::)} \mu_s + \left( \Sigma^{-1}_{s(t_1::)} \mu_s \right)^2 = \xi_{t_1,t_1} \right) \right) \left( \int_{0}^{\infty} \Pi(s(t)) g_s + \sum_{j=1}^{n} \left( \xi_{t_1,j} \right)^2 \left( \int_{0}^{\infty} \Pi(s(t)) \log(S_j) g_s \right) \right).
\]

The second-order mixed partial derivative of \( g_s \) is:
\[
\frac{\partial^2}{\partial S_{l_1} \partial S_{l_2}} g_s = gs \left( \frac{1}{S_{l_1} S_{l_2}} \right) \left( 1 + \left( \Sigma^{-1}_{s(l_1::)} \log(s) - \Sigma^{-1}_{s(l_1::)} \mu_s \right) + \left( \Sigma^{-1}_{s(l_2::)} \log(s) - \Sigma^{-1}_{s(l_2::)} \mu_s \right) - \xi_{l_1,l_2} \right)
\]
\[
= gs \left( \frac{1}{S_{l_1} S_{l_2}} \right) \left( 1 - \Sigma^{-1}_{s(l_1::)} \mu_s - \Sigma^{-1}_{s(l_2::)} \mu_s + \Sigma^{-1}_{s(l_1::)} \mu_s \Sigma^{-1}_{s(l_2::)} \mu_s - \xi_{l_1,l_2} \right) + \left( \Sigma^{-1}_{s(l_1::)} \log(s) \right) \left( 1 - \Sigma^{-1}_{s(l_2::)} \mu_s \right) + \left( \Sigma^{-1}_{s(l_2::)} \log(s) \right) \left( 1 - \Sigma^{-1}_{s(l_1::)} \mu_s \right)
\]
\[
+ \left( \Sigma^{-1}_{s(l_1::)} \log(s) \right) \left( \Sigma^{-1}_{s(l_2::)} \log(s) \right).
\]

Re-arranging we get:

2a. The term \( gs \left( \frac{1}{S_{l_1} S_{l_2}} \right) \left( 1 - \Sigma^{-1}_{s(l_1::)} \mu_s - \Sigma^{-1}_{s(l_2::)} \mu_s + \Sigma^{-1}_{s(l_1::)} \mu_s \Sigma^{-1}_{s(l_2::)} \mu_s - \xi_{l_1,l_2} \right) \) setting \( \alpha_1 = -1, \alpha_2 = -1, \alpha_i = 0, i \in \{1, \ldots, n\}, i \neq l_1 \neq l_2 \) and applying the Corollary (A.2), could be transformed as MVLN densities:
\[
\hat{g}_s = gs \left( \frac{1}{S_{l_1} S_{l_2}} \right) \left( 1 - \Sigma^{-1}_{s(l_1::)} \mu_s - \Sigma^{-1}_{s(l_2::)} \mu_s + \Sigma^{-1}_{s(l_1::)} \mu_s \Sigma^{-1}_{s(l_2::)} \mu_s - \xi_{l_1,l_2} \right)
\]
\[
= \exp(-\mu_t - \mu_2) \left( 1 - \Sigma^{-1}_{s(l_1::)} \mu_s - \Sigma^{-1}_{s(l_2::)} \mu_s + \Sigma^{-1}_{s(l_1::)} \mu_s \Sigma^{-1}_{s(l_2::)} \mu_s - \xi_{l_1,l_2} \right) g_s.
\]

2b. Log-contracts times MVLN densities,
\[
\exp(-rt) \left( \sum_{j=1}^{n} \left( \xi_{t_1,j} \right)^2 \left( \int_{0}^{\infty} \Pi(s(t)) \log(S_j) g_s \right) \right).
\]
2c. Cross log-contracts of second-order, that will appear when calculating cross-sensitivities of the MVLN density:

\[ g_S \left( \frac{1}{S_{l_1} S_{l_2}} \right) \left( \Sigma_{s_{l_1},(l_1,\cdot)}^{-1} \log (s) \Sigma_{s_{l_2},(l_2,\cdot)}^{-1} \log (s) \right). \]

Finally, the second-order mixed moment correction is:

\[ \exp(-rt) \sum_{l_1=1}^{n} \sum_{l_2=2}^{n} M_{l_1,l_2} \frac{1}{2} \left( \int_{0}^{\infty} \Pi(s(t)) \frac{\partial^2}{\partial S_{l_1}^2} g_S \right) = \exp(-rt) \]

\[ \left( \sum_{l_1=1}^{n} \sum_{l_2=1}^{n} M_{l_1,l_2} \frac{1}{2} \exp(-\mu_{l_1} - \mu_{l_2}) \left( 1 - \Sigma_{s_{l_1},(l_1,\cdot)}^{-1} \mu_s - \Sigma_{s_{l_2},(l_2,\cdot)}^{-1} \mu_s + \Sigma_{s_{l_1},(l_1,\cdot)}^{-1} \mu_s \Sigma_{s_{l_2},(l_2,\cdot)}^{-1} \mu_s - \varsigma_{l_1,l_2} \right) \right) \left( \int_{0}^{\infty} \Pi(s(t)) g_S \right) \]

\[ \left( \sum_{l_1=1}^{n} \sum_{l_2=1}^{n} M_{l_1,l_2} \frac{1}{2} \exp(-\mu_{l_1} - \mu_{l_2}) \sum_{j=1}^{n} \varsigma_{l_1,l_1} \left( 1 - \Sigma_{s_{l_1},(l_1,\cdot)}^{-1} \mu_s \right) + \varsigma_{l_2,l_2} \left( 1 - \Sigma_{s_{l_2},(l_2,\cdot)}^{-1} \mu_s \right) \right) \left( \int_{0}^{\infty} \Pi(s(t)) \log(S_{l_1}) g_S \right) \]

\[ \left( \sum_{l_1=1}^{n} \sum_{l_2=1}^{n} M_{l_1,l_2} \frac{1}{2} \exp(-\mu_{l_1} - \mu_{l_2}) \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \varsigma_{l_1,j_1} \varsigma_{l_2,j_2} \right) \left( \int_{0}^{\infty} \Pi(s(t)) \log(S_{j_1}) \log(S_{j_2}) g_S \right). \]
Table 9: Option prices with a MGEE of jump-diffusion processes. The risk-neutral density \( f_{X(t)} \) is generated by a 5-dimensional jump-diffusion process with parameters \( \lambda = 10, \delta_i = 0, \nu_i = 0.05, r = 0.10, t = 0.25, \sigma_i = 0.2, (S_1(0), \ldots, S_5(0)) = (35, 25, 20, 15, 5), i \in \{1, \ldots, 5\} \). The payoff of the basket option is \( \Pi(S(t)) = \left( \sum_{i=1}^{5} S_i(t) - K \right)^+ \) with \( K \in \{90, 100, 110\} \). The mean parameters of the auxiliary MVLN density \( g_{S(t)} \) are set by the arbitrage-free constraints, and the volatility parameters are set equal to the jump-diffusion process. The best approximation is highlighted.

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Table 10: Volatility values obtained from the minimisation of $h_3(\tilde{\sigma})$ following the calibration algorithm described in Section 6.1. The volatilities $\tilde{\sigma}_i$ are components of the volatility vector that correspond to each case of Table 13. As the strike price $K$ does not affect the minimisation algorithm, there are only 16 vectors corresponding to the 48 cases. The column $\hat{\sigma}_i$ represents the optimal volatility vector for the auxiliary MVLN distribution when the constraint of equal components is imposed.

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Table 11: Volatility values obtained from the minimisation of $h_4(\tilde{\sigma})$ following the calibration algorithm described in Section 6.1. The volatilities $\tilde{\sigma}_i$ are components of the volatility vector that correspond to each case of Table 14. As the strike price $K$ does not affect the minimisation algorithm, there are only 16 vectors corresponding to the 48 cases. The column $\hat{\sigma}_i$ represents the optimal volatility vector for the auxiliary MVLN distribution when the constraint of equal components is imposed.

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Table 12: Option prices with MGEE of jump-diffusion processes calibrating $h_{3}(\theta)$. The risk-neutral density $f_{X(t)}$ is generated by a 5-dimensional jump-diffusion process with parameters $\lambda = 10, \delta_{t} = 0, \nu_{i} = 0.05, r = 0.10, t = 0.25, \sigma_{i} = 0.2, (S_{1}(0), \ldots, S_{5}(0)) = (35, 25, 20, 15, 5), i \in \{1, \ldots, 5\}$. The payoff of the basket option is $\Pi(S(t)) = \left(\sum_{i=1}^{5} S_{i}(t) - K\right)^{+}$ with $K \in \{90, 100, 110\}$. The mean parameters of the auxiliary MVLN density $g_{3}(t)$ are set by the arbitrage-free constraints, and the volatility parameters are obtained by (20). The best approximation is highlighted.

<table>
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Table 13: Option prices with MGEE of jump-diffusion processes calibrating $h_3(\tilde{\sigma})$. The risk-neutral density $f_{X(t)}$ is generated by a 5-dimensional jump-diffusion process with parameters $\lambda = 10, \delta_i = 0, \nu_i = 0.05, r = 0.10, t = 0.25, \sigma_i = 0.2, (S_1(0), \ldots, S_5(0)) = (35, 25, 20, 15, 5), i \in \{1, \ldots, 5\}$. The payoff of the basket option is $\Pi(S(t)) = \left( \sum_{i=1}^{5} S_i(t) - K \right)$ with $K \in \{90, 100, 110\}$. The mean parameters of the auxiliary MVLN density $g_{S(t)}$ are set by the arbitrage-free constraints, and the volatility parameters are obtained by minimising $h_3(\tilde{\sigma}) = \|M_{t,\tilde{\sigma}}\|_2 + \|M_{t,\tilde{\sigma}}\|_2$. The best approximation is highlighted.

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The best approximation is highlighted.
Table 14: Option prices with MGEE of jump-diffusion processes calibrating $h_4(\tilde{\sigma})$. The risk-neutral density $f_{X(t)}$ is generated by a 5-dimensional jump-diffusion process with parameters $\lambda = 10$, $\delta_i = 0$, $\nu_i = 0.05$, $r = 0.10$, $t = 0.25$, $\sigma_i = 0.2$, $(S(0)) = (35, 25, 20, 15, 5)$, $i \in \{1, \ldots, 5\}$. The payoff of the basket option is $\Pi(S(t)) = \left( \sum_{i=1}^{5} S(t_i) - K \right)$ with $K \in \{90, 100, 110\}$. The mean parameters of the auxiliary MVLN density $g_{S(t)}$ are set by the arbitrage-free constraints, and the volatility parameters are obtained by minimising $h_4(\tilde{\sigma}) = \| M_{i,t} \|_2 + \| M_{i,t,2,t} \|_2 + \| M_{i,t,2,t,3} \|_2$.

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<th>$r$</th>
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Table 15: Percentage of the Monte Carlo simulated paths that are evaluated with the MGEE density and generate positive value. The columns Uncalibrated, \((h_2(\tilde{\sigma}),\text{MGEE})\), \((h_3(\tilde{\sigma}),\text{MGEE})\) and \((h_4(\tilde{\sigma}),\text{MGEE})\) correspond to the percentage of the 20,000,000 simulation paths that have positive values when they are evaluated under the MGEE density, of Tables 9, 12, 13, and 14, respectively.

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