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Multifactor Empirical Asset Pricing Under Higher-Order Moment Variations

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ABSTRACT

Even though an asset pricing model can be expressed in a classical Beta or in the relatively new stochastic discount factor (SDF) representation, their key empirical features - efficiency and robustness - may differ when estimated by the generalized method of moments. Using US and UK data we find that the SDF approach is more likely to be less efficient but more robust than Beta method. We derive the analytical asymptotic variance and show that the main drivers of this trade-off are the higher-order moments of the factors, in which skewness and covariance between returns and factors play an important role.

Keywords: Empirical Asset Pricing, Factor Models, Financial Econometrics, Generalized Method of Moments, Stochastic Discount Factor, Beta pricing, Efficiency.

JEL Classification: C51, C52, G12.

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I. Introduction

Any asset pricing model can be formally characterized either under a Beta or a stochastic discount factor (SDF) representation. The SDF representation states that the value of a financial asset equals the expected value of the product of the payoff on the asset and the SDF. Accordingly to the Beta representation, the expected return on an asset, is a linear function of its factor betas. The Beta approach is widely implemented in the finance literature using the two-stage cross-sectional regression methodology advocated by Black et al. (1972), Fama and MacBeth (1973), and Kan et al. (2013). The relatively new SDF characterization can be traced back to Dybvig and Ingersoll (1982), Hansen and Richard (1987), and Ingersoll (1987), who derive it for a number of theoretical asset pricing models.

Even though both characterizations are theoretically equivalent, the parameters of interest are in general dissimilar under the two setups. In particular, the Beta representation is formulated to analyze the factor risk premium δ, and the Jensen alpha α. In contrast, the SDF representation is intended to analyze the parameters that enter into the imposed stochastic discount factor λ, and the pricing errors π.[1] As a matter of fact, only when the factor is standardized to have zero mean and a variance equal to one, the parameters of interest coincide, $\delta = \lambda$, and $\alpha = \pi$; however this is rather a theoretical assumption than a real possibility.

The fact that both representations are equivalent, imply that there is a one-to-one mapping between δ and λ, and between α and π, which may facilitate the comparison of the estimators. Though, this theoretical equivalence does not necessarily entail an empirical equivalence. Therefore, the experimental questions that arise are: (1) *Is it better to make inferences on δ or on λ?* and analogously: (2) *Is it better to make inferences on α or on π?*

Our empirical and analytical results show that, in general, the Beta method is more **efficient** but less **robust** than the SDF method. That is, δ and π have lower simulated standard errors than their correspondent λ and α. We illustrate that the choice of the

[1]See Ferson and Jagannathan (1996) for a general approach to this equivalence.

Beta or the SDF method can be condensed into a trade-off between efficiency versus robustness. In this sense, we provide empirical motivated evidence about what drives this trade-off. This “easy way” to validate information is valuable for researchers and practitioners because they would have a *priori* rough idea about the benefits or the pitfalls of following either method in order to conduct empirical analysis.

To disentangle the source of the empirical results, we extend Jagannathan and Wang (2002) analytical results to the case of multifactor asset pricing models. We find that the source of the efficiency of the Beta method over the SDF method, is rooted in the higher-order moments effects on factors, that impact more the SDF than the Beta estimation.[2] We show that the factor negative skewness – which is usually a characteristic of the momentum portfolios, constitute a drawback to the SDF method at estimating λ even if the sample is as large as one thousand. Additionally, the application to the UK data illustrates that this relative disadvantage increases as we consider smaller samples. In any case, this drawback at estimating λ represents an advantage at estimating π given that the method is basically devoted to minimize the pricing errors. In this paper, we take advantage of the one-to-one mapping between Beta and SDF estimators in order to transform the Beta estimators into SDF units. By doing so, we are able to perform a fair comparison of the simulated standard errors because even though the values do not coincide numerically, they will have the same measure units.

Out study contributes to the asset pricing and financial econometrics literature, where it is a common trend to compare the performance of different econometric procedures within the Beta framework, or within the SDF method. For example, Jagannathan and Wang (1998) compare the asymptotic efficiency of the two-stage cross-sectional regression and the Fama-MacBeth procedure. Shanken and Zhou (2007), analyze the finite sample properties and empirical performance of the Fama-MacBeth maximum likelihood, and

[2] Graham and Harvey (2001) find in a survey that almost 74% of the respondents (Chief Financial Officers - CFOs) used “always or most of the times” the CAPM to calculate the cost of capital for their projects versus 40% of the CFOs that used the average historical returns of a stock. This second method – average historical returns of a stock – can be considered a particular version of the SDF method. Our results have an economic importance due to the estimation of the cost of capital, as every increase/decrease of 1% in the cost of capital due to the estimation variance, can be translated into an increased/decreased $1MM additional cost per every $100MM project valuation.

GMM for Beta pricing models. Other related examples can be found in Amsler and Schmidt (1985), Velu and Zhou (1999), Farnsworth et al. (2002), Chen and Kan (2004), Kan and Robotti (2008), and Kan and Robotti (2009), just to mention a few. However, only recently, there have been attempts to evaluate the performance in finite samples of the Beta versus the SDF approaches, and this is where our main interest relies because it has become a rich research field in financial econometrics and asset pricing.

In the first attempt to evaluate the performance in finite samples of the Beta versus the SDF approaches, using a standardized single-factor model, Kan and Zhou (1999) show that the SDF method is much less efficient than the Beta method. Jagannathan and Wang (2002), Cochrane (2001), and Cochrane (2005), debate this conclusion in a non-standardized single-factor model and assuming joint normality for both, the asset returns and the factors; they conclude that the SDF method is as efficient as the Beta method for estimating risk premiums. In addition, they find that the specification tests are equally powerful. Short after, in a non-published manuscript, Kan and Zhou (2001) show that, under a more general distributional assumption, and considering non-standardized factors, inference based on λ can still be less reliable than inference based on δ in realistic situations where the factors are leptokurtic. Contributing to the discussion, Ferson (2005) concludes that when the two methods correctly exploit the same moments they deliver nearly identical results. The interest about the topic has risen and attracted other researchers. For example, Lozano and Rubio (2011) show evidence suggesting that inference on δ and π, is more reliable than inference on their correspondent estimators λ and α. On the other hand, Peñaranda and Sentana (2015) show that a particular GMM procedure gives rise to numerically identical Beta and SDF estimates.

In this paper, we advocate that the core difference between the SDF and the Beta methods is a matter of trade-off between efficiency versus robustness. Commonly in econometrics, a robustness approach places fewer restrictions to the estimator. In this sense, the SDF method does not place any restriction to λ, whereas the Beta method explicitly incorporates the definition of δ as a subset of the moment restrictions. Therefore, it may be reasonable to expect that the Beta method should be more efficient than the SDF method and, conversely, the SDF method should be more robust than the Beta method.

Regarding the Beta method, there is no disagreement about which set of moments have to be included in the GMM estimation. However, there are two dominant and subtly different SDF representations in the literature. The first one defines the SDF as a linear function of the factors. However, Kan and Robotti (2008) point out that this is problematic because the specification test statistic is not invariant to an affine transformation of the factors. Therefore, in our study we also consider an alternative specification that defines it as a linear function of de-meaned factors. These two SDF variants are also known as un-centred and centred respectively. The methods are then applied to estimate and evaluate the single-factor CAPM of Sharpe (1964), Lintner (1965) and Mossin (1966); the three factor Fama and French (1993); the four factor Carhart (1997), and the three factor of Lozano and Rubio (2011) (RUH) models in an application to US data. We also provide some illustrative results using a reduced UK sample.

There are two more differences in our finite sample approach with respect to previous similar studies that raise a larger number of implications and original results. The first is that we assume factors and returns are drawn from their empirical distribution. The second is that we evaluate not only single, but multi-factor linear asset pricing models. By doing so, we allow the methods to perform under the presence of notably non-normal distributions, as it actually happens in more realistic situations. Furthermore, we use a diverse number of test assets in order to address the tight factor structure problem advocated by Lewellen et al. (2009).

The debate about the differences regarding both methodologies is still far from being conclusive. Furthermore, the available results in the current literature normally rely on the evaluation of the single-factor model which is still an important benchmark. In spite of this, multi-factor pricing models are by far the most commonly used in current empirical applications, and they incorporate factors with greater non-normalities than the market factor itself. For instance, in our sample, momentum factor has similar mean and variance as the market factor; nonetheless, it is almost three times more leptokurtic. Likewise, market factor is close to being symmetric, although skewness for size and value factors is positive, but markedly negative for momentum. Table I has the full descriptive statistics for factors and portfolios used in this paper.

[Place Table I about here]

The outline of the remainder of the paper is as follows. Section II presents the methodology, Section III presents the analytical results, Section IV presents the empirical results, while Section V concludes.

II. Methodology

In order to estimate and evaluate the Beta and SDF representations we follow the GMM econometric procedure of Hansen (1982). Therefore we would retrieve valid inferences even if the assumptions of serial independence, conditional homoskedasticity or normality are not fully realistic in practice. For a deeper discussion about the GMM applied to the estimation and evaluation of asset pricing models see Campbell et al. (1997), Jagannathan et al. (2002), and Cochrane (2005).

A. Beta method

We denote r_t as the vector of N stock returns in excess of the risk-free rate and f_t a vector of K economy-wide pervasive risk factors during period t. The mean and the covariance matrix of the factors are denoted by μ and $\mathbf{\Omega}$, where $\mu = \mathrm{E}[f_t]$, and $\mathbf{\Omega} = \mathrm{cov}(f_t)$. The standard linear asset-pricing model under the Beta representation is given by

$$\mathrm{E}[r_t] = \mathbf{B}\delta, \tag{1}$$

where δ is the vector of factor risk premiums, and $\mathbf{B}$ is the matrix of $N \times K$ factor loadings which measure the sensitivity of asset returns to the factors, defined as

$$\mathbf{B} \equiv \mathrm{E}[r_t(f_t - \mu)']\mathbf{\Omega}^{-1}. \tag{2}$$

Equivalently, we can identify $\mathbf{B}$ as a parameter in the time-series regression

$$r_t = \phi + \mathbf{B} f_t + \epsilon_t, \tag{3}$$

where the residual ϵ_t has zero mean and covariance $\mathbf{\Sigma}_{\epsilon_t}$, and it is uncorrelated with the factors f_t. We consider the general case where the factors might have higher-order moments, denoted by κ_3, the coskewness tensor, and κ_4, the cokurtosis tensor.[3] The specification of the asset-pricing model under the Beta representation in equation (1) imposes the following restriction on the time-series intercept, $\phi = (\delta - \mu)\mathbf{B}$. By substituting this restriction in the regression equation, we obtain:

$$r_t = \mathbf{B}\left(\delta - \mu + f_t\right) + \epsilon_t \quad \text{where:} \quad \begin{cases} \mathrm{E}\left[\epsilon_t\right] = 0_N \\ \mathrm{E}\left[\epsilon_t f_t'\right] = 0_{N \times K} \end{cases}. \tag{4}$$

Hence, the Beta representation in equation (1) gives rise to the factor model in equation (4). The associated set of moment conditions g of the factor model are:

$$\begin{aligned} &\mathrm{E}[r_t - \mathbf{B}(\delta - \mu + f_t)] = 0_N, \\ &\mathrm{E}[[r_t - \mathbf{B}(\delta - \mu + f_t)] f_t'] = 0_{N \times K}. \end{aligned} \tag{5}$$

However, when the factor is the return on a portfolio of traded assets, as in the single and multi-factor models analyzed in this paper – the CAPM, Fama-French, RUH, and Carhart factor models – it can be verified that the estimate of μ (the sample mean of the factor) is also the estimate of the risk premium δ. Therefore, given $\delta = \mu$, the moment conditions given in equation (5) simplify to:

$$\begin{aligned} &\mathrm{E}[r_t - \mathbf{B} f_t] = 0_N, \\ &\mathrm{E}[(r_t - \mathbf{B} f_t) f_t'] = 0_{N \times K}, \\ &\mathrm{E}[f_t - \mu] = 0_K, \end{aligned} \tag{6}$$

[3] Coskewness and cokurtosis have been addressed in risk premium asset pricing studies such as Harvey and Siddique (2000), Dittmar (2002) and Guidolin and Timmermann (2008). Tensors is a notation for N-dimensional arrays; the coskewness is a 3-dimensional array, and the cokurtosis is a 4-dimensional array.

where neither δ or μ appear in the first two restrictions of equation (6) but it is necessary to include the definition of μ to identify the vector of risk premiums δ as a third moment restriction. Nevertheless, it is also possible to estimate the last moment restriction of equation (6) outside the GMM framework by computing $\mu = E[f_t]$. This is because the number of added moment restrictions in equation (6) compared with equation (5) is the same as the number of added unknown parameters. Hence, the efficiency of equation (5) and equation (6) remains the same. By following this alternative, we drop the factor-mean moment condition without ignoring that it has to be estimated. An additional moment condition to estimate the variance $\mathbf{\Omega}$ could also be added to equation (6). However the variance can also be estimated outside the GMM framework without affecting efficiency.

Now, following the usual GMM notation, we define the vector of unknown parameters $\theta = [\text{vec}(\mathbf{B})' \quad \mu']'$, where the vec operator 'vectorizes' the $\mathbf{B}_{N\times K}$ matrix by stacking its columns, and the observable variables are $x_t = [r_t' \quad f_t']'$. Then, the function g in the moment restriction is given by:

$$g(x_t, \theta) = \begin{pmatrix} r_t - \mathbf{B}f_t \\ \text{vec}[(r_t - \mathbf{B}f_t)f_t'] \\ f_t - \mu \end{pmatrix}_{(N+NK+K)\times 1}, \tag{7}$$

in which, for any θ, the sample analogue of $\mathrm{E}[g(x_t, \theta)]$ is equal to

$$g_T(\theta) = \frac{1}{T}\sum_{t=1}^{T} g(x_t, \theta). \tag{8}$$

Then, a natural estimation strategy for θ is to choose the values that make $g_T(\theta)$ as close to the zero vector as possible. For that reason we choose θ to solve

$$\min_{\theta} g_T(\theta)'\mathbf{W}^{-1}g_T(\theta). \tag{9}$$

To compute the first-stage GMM estimator θ_1 we consider $\mathbf{W} = \mathbf{I}$ in equation (9). The second-stage GMM estimator θ_2 is the solution of equation (9) when the weighting matrix is the spectral density matrix of $g(x_t, \theta_1)$:

$$\mathbf{S} = \sum_{j=-\infty}^{\infty} \mathrm{E}[g(x_t, \theta_1) g(x_t, \theta_1)'], \tag{10}$$

where $\mathbf{S}$ is of size $N \times N$.

In order to examine the validity of the pricing model derived from the moment restrictions in equation (6) we can test whether the vector of N Jensen's alphas, given by $\alpha = \mathrm{E}[r_t] - \delta \mathbf{B}$ is jointly equal to zero.[4] This can be done using the J-statistic with an asymptotic χ^2 distribution. Given that there are $N + NK + K$ equations and $NK + K$ unknown parameters in equation (7), then the degrees of freedom is equal to N.

The covariance matrix of the pricing errors, $\mathrm{Cov}(g_T)$, is given by

$$\mathrm{Cov}(g_T) = \frac{1}{T}\left[(I - \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}')\mathbf{S}(I - \mathbf{B}(\mathbf{B}'\mathbf{B})^{-1}\mathbf{B}')\right], \tag{11}$$

and the test is a quadratic form of the vector of pricing errors. In particular, the Hansen (1982) J-statistic is computed as

$$\begin{aligned} &\text{First-stage:} && g_T(\theta_1)'\mathrm{Cov}(g_T)^{-1} g_T(\theta_1) \sim \chi^2_N, \\ &\text{Second-stage:} && T g_T(\theta_2)' S^{-1} g_T(\theta_2) \sim \chi^2_N. \end{aligned} \tag{12}$$

Both the first and second-stage statistics in equation (12) lead to the same numerical value. However, if we weight equations (11) and (12) by any other matrix different to $\mathbf{S}$, such as $\mathrm{E}[r_t r_t']$ or $\mathrm{Cov}[r_t]$, this result no longer holds.

B. The SDF method

To derive the SDF representation from the Beta representation we follow Ferson and Jagannathan (1996), and Jagannathan and Wang (2002). First, we substitute the expression for $\mathbf{B}$ in equation (2) into equation (1) and rearrange the terms, to obtain

$$\mathrm{E}[r_t] - \mathrm{E}[r_t \delta' \mathbf{\Omega}^{-1} f_t - r_t \delta' \mathbf{\Omega}^{-1} \mu'] = \mathrm{E}[r_t(1 + \delta' \mathbf{\Omega}^{-1} \mu - \delta' \mathbf{\Omega}^{-1} f_t)] = 0_N.$$

[4]This approach is known as the restricted test, see MacKinlay and Richardson (1991).

Again, if we are considering traded factors, then $\delta = \mu$ so $1 + \delta'\mathbf{\Omega}^{-1}\mu = 1 + \mu'\mathbf{\Omega}^{-1}\mu \geq 1$, then divide each side by $1 + \delta'\mathbf{\Omega}^{-1}\mu$,[5]

$$\mathrm{E}\left[r_t\left(1 - \frac{\delta'\mathbf{\Omega}^{-1}}{1 + \delta'\mathbf{\Omega}^{-1}\mu} f_t\right)\right] = 0_N.$$

If we transform the vector of risk premiums δ into a vector of new parameters λ as follows,

$$\lambda = \frac{\mathbf{\Omega}^{-1}\delta}{1 + \delta'\mathbf{\Omega}^{-1}\mu}, \tag{13}$$

then we obtain the following SDF representation, which is at the same time the set of moment restrictions g of the linear asset-pricing model,

$$\mathrm{E}[r_t(1 - \lambda' f_t)] = 0_N, \tag{14}$$

where the random variable $m_t \equiv 1 - \lambda' f_t$ is the SDF because $\mathrm{E}[r_t m_t] = 0_N$. Alternatively, we could derive the Beta representation from the SDF representation by expanding m and rearranging the terms.

From the moment restrictions, equation (14), we obtain the vector of N pricing errors defined as $\pi = \mathrm{E}[r_t] - \mathrm{E}[r_t f_t']\lambda$. The analytical solution of equation (14) is obtained by GMM.[6] Writing the sample pricing errors as:

$$g_T(\lambda) = -\mathrm{E}[r_t] + \mathrm{E}[r_t f_t']\lambda, \tag{15}$$

define $\mathbf{D}^U = -\frac{\partial g_T(\lambda)}{\partial \lambda'} = \mathrm{E}[r_t f_t']$, the second-moment matrix of returns and factors. The first-order condition to minimize the quadratic form of the sample pricing errors, equation (9), is $-\left(\mathbf{D}^U\right)' \mathbf{W}[\mathrm{E}[r_t] - \mathbf{D}^U \lambda'] = 0$, where $\mathbf{W}$ is the GMM weighting matrix of size $N \times N$, equal to the identity matrix in the first-stage estimator and equal to the spectral density matrix $\mathbf{S}$, equation (10), in the second-stage estimator. Therefore, the

[5] Even when the factors are not traded, it is common to suppose $1 + \delta'\mathbf{\Omega}^{-1}\mu \neq 0$.

[6] This is useful given the need to undertake vast numbers of simulations. Similar simplifications of multi-dimensional optimization problems for Beta models can be found in Shanken and Zhou (2007).

GMM estimates of λ are:

$$\begin{aligned}
\widehat{\lambda}_1^U &= \left((\mathbf{D}^U)' \, \mathbf{D}^U \right)^{-1} (\mathbf{D}^U)' \, \mathrm{E}[r_t], \\
\widehat{\lambda}_2^U &= \left((\mathbf{D}^U)' \, \mathbf{S}^{-1} \mathbf{D}^U \right)^{-1} (\mathbf{D}^U)' \, \mathbf{S}^{-1} \mathrm{E}[r_t].
\end{aligned} \tag{16}$$

For illustrative purposes, we decorate with an U to $\widehat{\lambda}$ to indicate that the estimator comes from the un-centred specification and with a C to indicate that it comes from the centred specification.

Specifying the SDF as a linear function of the factors as in equation (14) has been very popular in the empirical literature. However, Kan and Robotti (2008) point out that this is problematic because the specification test statistic is not invariant to an affine transformation of the factors; Burnside (2007) also reaches to similar conclusions. Therefore, we also consider an alternative specification that defines the SDF as a linear function of de-meaned factors. Examples of this representation can be found in Julliard and Parker (2005) and Yogo (2006).

The alternative centred version of equation (14) is defined as:

$$\mathrm{E}[r_t[1 - \lambda(f_t - \mu)]] = 0_N. \tag{17}$$

According to Jagannathan and Wang (2002) and Jagannathan et al. (2008), it is also possible to estimate μ in equation (17) outside of the GMM estimation by computing $\mu = \mathrm{E}[f_t]$. This is because the number of added moment restrictions is the same as the number of added unknown parameters. Hence, the efficiency of the estimators remains the same. By following this alternative, we can drop the factor-mean moment condition without ignoring that it has to be estimated, to obtain analytical expressions for $\widehat{\lambda}_1^C$ and $\widehat{\lambda}_2^C$.

Naturally, the procedure to solve the moment restrictions in equation (17) is similar to that for the un-meaned SDF^U method. In particular, we substitute $\mathrm{E}[r_t f_t]$ for $\mathrm{Cov}[r_t f_t]$ in equation (15), then define $\mathbf{D}^C = -\frac{\partial g_T(\lambda)}{\partial \lambda'}$ as the covariance matrix of returns and factors.

Finally, the SDF^C first and second stage GMM estimates are

$$
\begin{aligned}
\widehat{\lambda}_1^C &= \left((\mathbf{D}^C)' \, \mathbf{D}^C \right)^{-1} (\mathbf{D}^C)' \, \mathrm{E}[r_t], \\
\widehat{\lambda}_2^C &= \left((\mathbf{D}^C)' \, \mathbf{S}^{-1} \mathbf{D}^C \right)^{-1} (\mathbf{D}^C)' \, \mathbf{S}^{-1} \mathrm{E}[r_t].
\end{aligned} \tag{18}
$$

The specification tests can be conducted by following equations (8) and (12), the only difference being that we substitute $\mathbf{B}$ by $\mathbf{D}^U = \mathrm{E}[r_t f_t]$ (the second moment matrix of returns and factors) for the SDF^U case, and by $\mathbf{D}^C = \mathrm{Cov}[r_t f_t]$ (the covariance matrix of returns and factors) for the SDF^C case. The degrees of freedom in equation (12) are specific for the Beta method, in the SDF method the degrees of freedom is equal to $N-K$, since there are N equations and K unknown parameters in both equations (14) and (17).

Equations (11) and (12) are weighted by equation (10), since it is statistically optimal. This approach was first suggested by Hansen (1982) as it maximizes the asymptotic statistical information in the sample about a model, given the choice of moments. However, there are also alternatives for this weighting matrix which are suitable for model comparisons because they are invariant to the model and their parameters. For instance, Hansen and Jagannathan (1997) suggest the use of the second moment matrix of excess returns $\mathbf{W} = \mathrm{E}[r_t r_t']$ instead of $\mathbf{W} = \mathbf{S}$. Also, Burnside (2007), Balduzzi and Yao (2007), and Kan and Robotti (2008) suggest that the SDF^C method should use the covariance matrix of excess returns $\mathbf{W} = \mathrm{Cov}[r_t]$. We investigate the implications of using these alternative weighting matrices.

C. *Comparison of the methods*

There is a one-to-one mapping between the factor risk premium δ and the SDF parameter λ, which facilitates the comparison of the two methods.[7] Hence, we can derive an estimate of λ not only by the SDF method but also by the Beta method. By the same token we can derive an estimate of δ not only by the Beta method but also by the SDF

[7] Thanks to Raymond Kan for kindly sharing complementary econometric notes on Kan and Zhou (2001).

method. From the previous definition of λ in equation (13), we have:

$$\lambda = \delta' \left(\mathbf{\Omega} + \delta\mu'\right)^{-1}, \qquad \text{or} \qquad \delta = \frac{\mathbf{\Omega}\lambda}{1-\mu'\lambda}. \tag{19}$$

In a similar way, by substituting the equation (19) into π, we can find a one-to-one mapping between π from the SDF method and α from the Beta method.

$$\pi = \frac{\mathbf{\Omega}}{\mathbf{\Omega} + \delta\mu'}\alpha, \qquad \text{or} \qquad \alpha = \frac{\mathbf{\Omega} + \delta\mu'}{\mathbf{\Omega}}\pi. \tag{20}$$

We cannot directly compare λ versus δ, neither π versus α since they are measured with different units. Any direct comparison would be biased simply for a scaling issue, and not necessarily for an intrinsic difference among the Beta and SDF methods. An alternative for avoiding this scaling issue is to transform δ into λ units, and α into π units following equations (19) and (20). For convenience, we will decorate all Beta estimators with '$*$' in order to identify that they are Beta estimators.

In the first formal attempt to compare both methods, Kan and Zhou (1999) assume that the factor has zero mean and unit variance, that is $\mu = 0$ and $\mathbf{\Omega} = \mathbf{1}$. In this standardized single factor model, equations (19) and (20) simplify to $\lambda = \delta$ and $\pi = \alpha$. By assuming that the mean and the variance of the factor are predetermined without estimation, they ignore the sampling errors associated with the estimates of μ and $\mathbf{\Omega}$ and conclude that the estimates of the Beta method are more efficient. Jagannathan and Wang (2002) and Cochrane (2001) explain the effects of standardized factors, showing that in general, predetermining the factor moments reduces the sampling error of the estimate in the Beta method and not in the SDF method.

However, with the Beta moment restrictions, equation (6), we only can make inference on δ, not on λ. Yet to compare the methods using equation (19) requires an estimator of $\mathbf{\Omega}$. One solution is to add an additional moment condition to equation (6) to estimate $\mathbf{\Omega}$. An alternative is to estimate μ and $\mathbf{\Omega}$ outside the GMM estimation. In simulation results not shown here, we find that the efficiency of both alternatives is the same. Hence we elect to estimate $\mathbf{\Omega}$ outside the GMM estimation.

Predetermining the values of μ and $\mathbf{\Omega}$ to be known constants, not necessarily $\mu = 0$ and $\mathbf{\Omega} = 1$, gives an informational advantage to the Beta method in terms of efficiency. Predetermining without estimation implies ignoring the sampling errors associated with μ^* and $\mathbf{\Omega}^*$, as a consequence λ^* becomes considerably more efficient than if we follow equation (6). In our simulation analysis, we consider the case where μ and $\mathbf{\Omega}$ must be estimated.

To summarize, the Beta method gives the GMM estimate δ^* while the SDF method gives the GMM estimate λ. In our simulation results, we transform the estimate δ^* into an estimate of λ and then compare the variances of the sampling distribution of $\widehat{\lambda}^*$ and $\widehat{\lambda}$. In the same way, we transform α^* into an estimate of π and then compare the efficiency of $\widehat{\pi}^*$ and $\widehat{\pi}$. We also compare the distributions of Hansen (1982) test of overidentification using the J-statistic of the transformed Beta $\widehat{J}^*$ and $\widehat{J}$ from the SDF method. The null hypothesis is that all pricing errors are zero.

III. Analytical results

In this Section we derive an general extension to the Jagannathan and Wang (2002) model, where the factor f_t is multivariate and can be non-Gaussian. The empirical results of the trade-off between efficiency and robustness, when using the Beta method and the SDF method for estimation, will provide an adequate framework for discussion. Nevertheless, having a theoretical solid quantitative framework will provide better insights of the empirical results obtained.

PROPOSITION 1: *Let f_t represent the multivariate pervasive risk factors, and consider the Beta representation in equation (1), and the SDF representation in equation (14). Then, the asymptotic variance of the SDF model $\hat{\lambda}$ parameters is,*

$$Avar(\hat{\lambda}) = \left((\mathbf{\Omega} + \mu\mu')' \mathbf{B}'\right)^{-1} \left(\frac{1}{a_{\epsilon_t}}\mathbf{\Sigma}_{\epsilon_t}^{-1} - \frac{1}{a_{\epsilon_t}^2}\mathbf{\Sigma}_{\epsilon_t}^{-1}\mathbf{B}\left(\mathbf{A}_B^{-1} + \frac{1}{a_{\epsilon_t}}\mathbf{B}'\mathbf{\Sigma}_{\epsilon_t}^{-1}\mathbf{B}\right)^{-1}\mathbf{B}'\mathbf{\Sigma}_{\epsilon_t}^{-1}\right) \times \left(\mathbf{B}\left(\mathbf{\Omega} + \mu\mu'\right)\right)^{-1}. \quad (21)$$

where a_{ϵ_t} and $\mathbf{A}_B$ are a scalar and a positive definite matrix function of the risk premium (see appendix A), and the asymptotic variance of the Beta λ^* parameter is equal to,

$$Avar(\lambda^*) \quad = \quad (\mathbf{\Omega} + \mu\mu')^{-1} S_b \left((\mathbf{\Omega} + \mu\mu')^{-1} \right)'. \tag{22}$$

where S_b is the covariance matrix of $g_b(f_t, \lambda_b) = (f_t(1 - \lambda_b f_t))$, and $\lambda_b = \mu' (\mathbf{\Omega} + \mu\mu')^{-1}$. In the case of single-factors, corresponding SDF and Beta GMM estimated risk-premium asymptotic variances are,

$$Avar(\hat{\lambda}) = \frac{\sigma^2(\sigma^2 + \mu^2)}{(\sigma^2 + \mu^2)^4} \left(\beta' \mathbf{\Sigma}_{\epsilon_t}^{-1} \beta \right)^{-1} + \frac{\sigma^2(\sigma^4 + \mu^4)}{(\sigma^2 + \mu^2)^4} + \frac{\kappa_4 \mu^2 + 2\kappa_3(\mu^3 - \mu\sigma^2) - 3\mu^2\sigma^2}{(\sigma^2 + \mu^2)^4}, \tag{23}$$

and,

$$Avar(\lambda^*) \quad = \quad \frac{((\sigma^2 + \mu^2) - 2\lambda_b E[f_t^3] + \lambda^2 E[f_t^4])}{(\sigma^2 + \mu^2)}, \tag{24}$$

with

$$\begin{aligned} E[f_t^3] \quad &= \quad \kappa_3 + 3\sigma^2\mu + \mu^3, \\ E[f_t^4] \quad &= \quad \kappa_4 + 4\kappa_3\mu + 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4. \end{aligned}$$

The Beta GMM estimated risk-premium asymptotic variance (24), can be approximated using the Delta method by,

$$Avar(\lambda^*) = \frac{\sigma^2(\sigma^2 + \mu^2)}{(\sigma^2 + \mu^2)^4} \left(\beta' \mathbf{\Sigma}_{\epsilon_t}^{-1} \beta \right)^{-1} + \frac{\sigma^6}{(\sigma^2 + \mu^2)^4}. \tag{25}$$

Corresponding SDF and Beta pricing errors asymptotic variance is equal to

$$Avar\,(\pi^*) = \left((\mathbf{\Sigma}_{\epsilon_t} + \delta\mu')\, \mathbf{\Sigma}_{\epsilon_t}^{-1} (\mathbf{\Sigma}_{\epsilon_t}^{-1})' \,(\mathbf{\Sigma}_{\epsilon_t} + \delta\mu') \right) Q \left(S_b - D_b(D_b' S^{-1} D_b) D_b' \right) Q', \tag{26}$$

and,

$$Avar\left(\hat{\pi}\right) = S_s - D_s\left(D_s' S_s^{-1} D_s\right) D_s'. \tag{27}$$

where

$$Q = \left[I_n, \mathbf{0}_{n\times n}, -\mathbf{B}, \boldsymbol{0}_{n\times 1}\right], \tag{28}$$

$$S_s = \mathbf{B}\mathbf{A}_B\mathbf{B}' + a_{\epsilon_t}\boldsymbol{\Sigma}_{\epsilon_t}, \tag{29}$$

$$D_s = -\mathbf{B}\left(\boldsymbol{\Omega} + \mu\mu'\right. \tag{30}$$

Proof. See appendix A. □

COROLLARY 1: *The asymptotic variance of the risk premium of the Beta method, $Avar(\hat{\lambda})$, in the case where the Delta method provides an accurate approximation, will be lower than the asymptotic variance of the SDF method, $Avar(\lambda^*)$, considering the stylized facts of the market returns – heavy tailed and negative skewed returns. In the case the risk factors, f_t, have a multivariate Gaussian distribution, the Beta method still has a lower asymptotic variance compared to SDF, but with an almost imperceptible difference.*

Proof. By observing the SDF and the Beta (Delta approximated) GMM risk-premiums asymptotic variances, equation (23) and equation (25), and after some algebra, we observe that their difference is equal to,

$$Avar(\hat{\lambda}) - Avar(\lambda^*) = \frac{\sigma^2(\mu^4)}{(\sigma^2+\mu^2)^4} + \frac{\mu^2\left(\kappa_4 - 2\kappa_3(\frac{\sigma^2-\mu^2}{\mu}) - 3\sigma^2\right)}{(\sigma^2+\mu^2)^4}. \tag{31}$$

Consider the following conditions based on returns stylized facts: $\mu < \sigma$ (volatility higher than expected returns), $3\sigma^2 < \kappa_4$ (heavy tailed returns), and $\kappa_3 < 0$ (negative skewed returns), then the the term (31) is positive and the result is yield. □

By analyzing equation (3), and considering that the error ϵ_t will have – in general – a standard normal distribution, we note the source of non-normality of the returns will have its origins in the non-normality of the factors, but even in the general case that we consider that ϵ_t might have higher-order moments, by results in Proposition 1, only the higher-order moments of f_t are important in the estimation.

Results of Corollary 1 depend on the assumptions of the Delta approximation. Nevertheless, in an empirical case, factors might have a much richer conditions. We study such conditions in the next section.

IV. Empirical Results

Researchers who estimate these kinds of asset pricing models are frequently faced with data sets of finite, and occasionally rather small, sample sizes.[8] In the case of the US, the data set can be as large as 1,100 monthly historical observations, but in the case of the UK, we can end up with about 450. It is therefore imperative to obtain a sense of the small sample performance of the two methods. Since finite-sample analytical results can be obtained only under certain distributional assumptions of the returns, factors and errors, it is customary to resort to some simulation technique, which allows to alter the simulation input and develop an understanding of how sensitive the results are with respect to the various features of the data generating process.

We use Bootstrap simulation[9] to find out whether the GMM estimators and test statistics have any bias. In particular, we are interested in evaluating the standard deviation of $\widehat{\lambda}^*$, $\widehat{\lambda}$, $\widehat{\pi}^*$, $\widehat{\pi}$, denoted as $\sigma(\cdot)$ and also the tail of the J-statistic distribution to conduct specification tests. We assume that the factors f_t are drawn from their empirical distribution which allows for non-normalities, autocorrelation, heteroskedasticity and non-independence of factors and residuals.

To artificially generate the excess returns we use the factor model, equation (4) where $t = 1, ..., T$. We consider two applications:(i) a Monte Carlo simulation, where the returns are generated by adding a multivariate normal error to the actual observed empirical factors, and (ii) a empirical simulation, where the returns are generated by bootstrapping

[8]See Ahn and Gadarowski (2004) for an examination of finite-sample properties of several model tests methods.

[9]Simulations were executed using the THOR Grid computational cluster provided by the Department of Economics,Finance and Accounting of the Maynooth University.

the observed historical returns, and factors are generated by bootstrapping the observed historical factors. The first application is provided to a numerically test the analytical asymptotic variance results, while the second application serves to measure the asymptotic variance of the different multivariate factors models analyzed.

For T, we consider the following four time horizons: 60, 600, 2000 and 3000 months for the first application (Monte Carlo simulation) and 60, 360, 600 and 1000 months for the second application (Bootstrapping empirical simulation). As Shanken and Zhou (2007) argue, varying T is useful in order to understand the small-sample properties of the tests and the validity of asymptotic approximations. For instance, we elect to examine a 5 year window since this may show how distorted results from taking a really small sample could potentially be, and also it is a commonly adopted horizon when using rolling windows, a 30 year window corresponds approximately to the sample sizes of Fama and French (1992, 1993) and Jagannathan and Wang (1996) while the 600 month sample matches the largest sample examined by Jagannathan and Wang (2002). We also examine 1000 months since this approximates the current size of the largest sample available on the Kenneth French's library [January 1927 to December 2018 – 1104 months], and could be considered as an approximation of the asymptotic variance. The estimators and specification tests are then calculated based on the T samples of the factors and returns generated from the factor model. We repeat this independently to obtain 10,000 draws of the estimators of λ, π (the pricing errors) and J (the overidentifying restriction statistic).

Previous related empirical studies such as Kan and Zhou (1999, 2001), Jagannathan and Wang (2002) and Cochrane (2005) focus on the CAPM model to test the efficiency of Beta and SDF methods. In this sense, our contribution is to evaluate the methods on multi-factor models in order to check for consistency in presence of other more leptokurtic factors commonly used by researchers. Therefore, we evaluate the two methods by estimating and testing the CAPM, Fama and French, RUH and the Carhart four factor models. We denote the factors as the excess market return (RMRF), size (SMB), value (HML) and momentum (UMD).[10]

[10] See Fama and French (1993), for a complete description of these factors.

In order to generate the excess returns from equation (4) we first need the $N \times K$ matrix $\mathbf{B}$, capturing the sensitivity of returns to the factor(s). This $\mathbf{B}$ matrix previously defined in equation (2), represents the slope coefficients in the OLS regressions of each N-test portfolio and K-factor model. We use three values of N to generate $\mathbf{B}$, these are the value weighted returns of the 10 size-sorted portfolios, the 25 Fama-French portfolios (the intersections of the 5 size and 5 book-to-market portfolios) and the 30 industry portfolios. As Lewellen et al. (2009) suggest, the traditional tests portfolios used in empirical work such as the size and 25 size/value sorted portfolios frequently present a strong factor structure, hence it seems reasonable to adopt other criteria (industry) for sorting.

In Table I we report the descriptive statistics of historical observations of factors and test portfolios, these values are used to calibrate the simulations of the two applications (Monte Carlo and Empirical), and Tables II and III provide details about the parameters used in the first application (Monte Carlo simulation). As can be seen from the four moments shown, the factors associated with the multi-factor models are quite different from the excess market return factor, in particular the momentum factor is almost three times more leptokurtic than the excess market return. Thus, it is important to consider an alternative which captures properties more consistent with the data such as excess kurtosis. Similar studies such as Kan and Zhou (2017) consider the Student-t distribution; however, the magnitude of kurtosis is still limited for a t-distribution with a finite fourth moment.[11] In previous simulations not shown here, a Student-t distribution with five degrees of freedom implies a kurtosis of 6 for the RMRF factor, which is still much lower than the empirical value of 11. Therefore, we consider the empirical distribution as the alternative to the multivariate normal.

[Place Table II about here]

[Place Table III about here]

We find that the choice of following either the Beta or the SDF method to empirically

[11]The asymptotic distribution theory for the GMM requires that returns and factors have finite fourth moments. Hence, there must be more than four degrees of freedom.

estimate and evaluate an asset pricing model can be traduced into a choice between efficiency or robustness. In particular, we show that frequently the Beta method dominates in terms of efficiency whereas the SDF method dominates in terms of robustness. Perhaps one of the most influential and cited papers in this field is the one of Jagannathan and Wang (2002), they argue that both methods lead to estimates with similar precision even in finite samples. Here, we illustrate that the conclusion of Jagannathan and Wang (2002) are only valid under very specific conditions that cannot be generalized.

A. Monte Carlo Simulation: Covergence

Figures 1a, 1b, 1c, and 1d present the asymptotic variance results from estimating λ^* with the GMM with the Beta method and λ_2^U with the GMM with the SDF method. The Monte Carlo simulation parameters used are in Tables II and III correspondingly. We can observe that the asymptotic variance when using the Beta method ($Avar(\lambda^*)$) is always lower than the asymptotic variance when using the second-stage non-centered SDF method ($Avar(\lambda_2^U)$).[12]

[Place Figure 1 about here]

Figure 1 results show that, in the case of a single-factor model using the market risk factor, where the skewness is closer to zero and the kurtosis is the lower of all single-factor models considered in this application, the Beta and the SDF Monte Carlo simulations and the Beta and SDF analytic estimated asymptotic variances converge towards the same value, consistent with Jagannathan and Wang (2002). Nevertheless, in the case of the size, value, and momentum single-factor models, the SDF estimated asymptotic variance is always higher than the Beta estimated: the higher third-order cumulant (κ_3) produces an increase in the asymptotic variance, consistent with analytic results in the Appendix A.

[12]Results for the first-stage SDF ($Avar(\lambda_1^U)$) and first-, and second-stage centered methods ($Avar(\lambda_1^C), Avar(\lambda_2^C)$) are not reported, but are higher than the second-stage non-centered SDF method ($Avar(\lambda_2^U)$).

B. Empirical application: Comparison of λ estimators

Tables IV and V compare the performance of Beta and SDF methods at estimating λ by the CAPM model using US data. Here, our results are qualitative and quantitative similar to those on Jagannathan and Wang (2002). In particular, Table IV shows that the expected value and the standard error of $\widehat{\lambda}^*$ and $\widehat{\lambda}$ are quite similar. Actually, the standard error of Beta estimators is statistically equal to the standard error of SDF estimators in most of the cases. Then, under this specific framework there are virtually no differences in terms of efficiency between the Beta and SDF methods. One of the main implications of this results is that there are no significant advantages to applying the Beta method to nonlinear asset pricing models formerly expressed in SDF representation through linear approximations.

[Place Table IV about here]

Table IV allows us to compare $\sigma(\widehat{\lambda}^*)$ versus $\sigma(\widehat{\lambda})$ instead of $\sigma(\widehat{\delta})$ versus $\sigma(\widehat{\lambda})$ in order to avoid misleading conclusions driven by a scaling issue. However, if we consider the possibility of intrinsic differences among the methods, the expected values of $\widehat{\lambda}^*$ and $\widehat{\lambda}$ do not necessarily would be similar in general. For this reason it is convenient to compute ratios of relative standard errors such as $\sigma_r(\lambda) = \sigma(\lambda) \div \mathrm{E}(\lambda)$. By doing so, we would have an accurate measure of the relative efficiency of the methods. These results are presented in Table V.

[Place Table V about here]

Ratios close to the unity represent a high degree of similarity of the efficiency of both methodologies at estimating λ. Higher ratios suggest that following the Beta method is better even in finite samples than following the SDF method to make inferences on λ estimators. Given that one is generally interested in testing the null hypothesis that the estimator is statistically equal to zero, the ratios such as the ones in Table V offer a good indication of which method leads to more accurate inferences. In particular, all values of Table V are slightly greater than one, which means that the Beta method leads to lower standard errors when estimating the CAPM. In general, the values of the ratios diminish as we increase the size of the sample and second-stage SDF estimators are indeed more efficient than first-stage estimators as expected. In addition, the un-centred SDF specification reveals a marginal advantage compared with the centred specification probably due to the extra moment restriction.

Our original contribution regarding the comparison of $\sigma_r(\lambda)$ is on the evaluation of multi-factor asset pricing models. Tables VI, VII and VIII show the ratios of relative standard errors of the Beta and SDF methods for the Fama-French, RUH and Carhart models respectively.

[Place Table VI about here]

[Place Table VII about here]

[Place Table VIII about here]

The correspondent expected values and standard errors for the multi-factor asset pricing models are in Tables IX, X and XI. We begin by describing the results for the Fama-French model, which is loaded with the market factor, the size factor and the value factor.

[Place Table IX about here]

[Place Table X about here]

[Place Table XI about here]

The first (upper) panel of Table VI is comparable to the Table V because in both cases the estimated λ corresponds to the market factor. Thus, it is not surprising to find a similar pattern which reinforces the conclusion that there are no significant differences when estimating the parameter associated with the market factor. The un-centred SDF method is again more efficient than the centred SDF method at estimating λ. And this time the second-stage un-centred method is marginally more efficient than the Beta method for samples smaller than one thousand.

Contrary to the market factor case, the standard error of Beta estimators linked to the size and value factors are statistically smaller than the correspondent standard errors of SDF estimators. This becomes evident in the higher ratios of second and third panels of Table VI. These results suggest that the empirical equivalence of both methods is subject to the loaded factor in the asset pricing model. In particular, the market factor does not represent a challenge to the SDF method whilst the value factor can lead to significant differences according to the $\sigma_r(\lambda)$ ratios. For instance, the relative standard error of the un-centred first-stage SDF method can be more than twice as big than the relative standard error of the Beta method. Beta estimators are even more efficient than second-stage SDF estimators, which by construction are intended to increase the estimation efficiency of $\widehat{\lambda}$.

The second multi-factor asset pricing model is the RUH, which factors are market, momentum and value. The estimation of the RUH model allows us to compare the efficiency of the estimator associated with the market and value factors in previous tables and introduces the result for momentum.

The ratios of relative standard errors $\sigma_r(\lambda)$ linked to the market factor are slightly lower than one; however the standard errors $\sigma(\lambda)$ are statistically equal for most of the cases. On the other hand, the magnitudes for the value factor are similar to the ones of the Fama-French model in Table VI. The second panel of Table VII shows the ratios for the momentum factor, which are somewhat greater than the ratios for value factor. Here, we also find that the SDF method may have difficulties in small samples which is reflected in values of 20.7 and 10.5.

The third and last multi-factor asset pricing model estimated is the Carhart model, which factors are market, size, value and momentum. The estimation of Fama-French, RUH and Carhart models represent the core contribution to the field, which lead to the main original results. However, we also present results for other test portfolios and to reduced sample sizes such as the ones researchers may face at evaluating models using UK data samples. For now, we show the relative standard errors of Carhart model λ estimators in Table VIII.

The results for the Carhart model support the argument that the efficiency of the method is associated with the factor. The lower ratios of relative standard errors of λ estimators are those linked to the size factor, followed by the market, value, and the greater are those of momentum factor. If the methods were empirical equal efficient, these ratios should be similar. However this is not the case, the relative efficiency is related with the associated factor of each estimator. Table VIII, suggest that SDF method marginally dominates Beta method in first and second panels which correspond to the results associated with the market and size factors, whilst Beta method outperform SDF method for the third and fourth panels which are related with the value and momentum factors. In sum, the previous tables show that the relative efficiency of the methods are not independent of the factors.

One may argue that results on Tables VI, VII and VIII may be partially driven by the added factors, and not because of any intrinsic difference among the methods. In order to address this possibility, we estimate four alternative single-asset pricing models. In each alternative, the model is loaded with a different factor namely market (CAPM), size, value and momentum. The ratios of relative standard errors of the Beta and SDF methods for these single-asset pricing models are on Table XII. The correspondent expected values and standard errors are in Table XIII.

[Place Table XII about here]

[Place Table XIII about here]

According to Table XII, the Beta method leads to more accurate inferences than the SDF method. The relevance of Table XII is that we artificially fix every element in the evaluation of the asset pricing model except the loaded factor. Given that ratios across panels are different, the statistical characteristics of each factor as well as their relation to the test portfolios are presumably the main drivers behind the differences of the methods at estimating λ.

C. Results: Robustness

In this subsection we turn our attention to the pricing errors estimates $\widehat{\pi}$, $\widehat{\pi}^*$, their corresponding standard errors $\sigma(\widehat{\pi})$, $\sigma(\widehat{\pi}^*)$ and their associated $\widehat{J}$, and $\widehat{J}^*$ statistics. The method would be more robust than the other if their simulated standard error of the pricing errors is lower.

To better understand where the advocated trade-off between efficiency and robustness comes from, we need to briefly describe the set of moments for each method. The traditional Beta GMM restrictions incorporate three sets of moments: (1) the N asset pricing restriction which define the α vector; (2) the $N \times K$ zero covariance between errors and factors; and (3) the K definition of δ, which is equal to the mean of the traded factor. Therefore, by imposing the definition of δ, the Beta method increases their relative estimation efficiency. On the other hand, the SDF method only incorporates the N asset pricing restriction for the un-centred specification which defines π; and the N asset pricing restriction plus the K mean of the factor for the centred specification. However, this method does not impose the definition of λ. As a result, it allows for freely varying its parameters in order to achieve lower pricing errors, favoring the robustness rather than the efficiency. By the same token, the specification test in the Beta method tends to under-reject in finite samples while the SDF method has approximately the correct size.

We begin by showing Table XIV which present the relative standard errors for the single and multi-factor models. Table XV present the values used in the construction of Table XIV.

[Place Table XIV about here]

[Place Table XV about here]

V. Conclusion

The interest in learning about the asymptotic and finite sample properties of asset pricing model estimators, like risk premiums and pricing errors, have attracted the attention of researchers for decades. This interest is motivated for an extensive list of theoretical and empirical applications mainly – but not exclusive – in economics and finance areas. It is not uncommon to find examples in which different econometric approaches involve a trade-off between efficiency and robustness since more efficient estimators may usually come at the cost of higher pricing errors and vice versa. However, as far as to our knowledge, this is the first time in which such analogy is explicitly used to better understand the differences of the Beta and SDF methods. This evidence is useful for researchers and practitioners because they could choose a proper procedure in terms of a given application. Consequently, the adequate selection leads to more accurate hypothesis tests, and other kinds of computations.

We argue that current studies which compare both approaches are conducted under certain conditions that are not sufficient to differentiate their performance. Once we relax those conditions, we show that differences between the two methods unsurprisingly emerge. Specifically, we find evidence suggesting that the Beta method leads to better risk premium estimators while the SDF method leads to better pricing error estimators in terms of efficiency. We evaluate the magnitude of the resulting biases and we also illustrate what are the main elements that drive such differences.

The debate about the studied methods is still open. Presumably, given the response of Jagannathan and Wang (2002) to Kan and Zhou (1999), the dominant trend is to consider that the methods do not exhibit significant differences in terms of efficiency. For example, see Cochrane (2001), Smith and Wickens (2002), Cochrane (2005), Nieto and Rodríguez (2005), Vassalou et al. (2006), Wang and Zhang (2006), Balvers and Huang (2007), Jagannathan et al. (2008), Cai and Hong (2009), and Brandt and Chapman (2018). Our results suggest that this similarity between the two methods only holds under very specific conditions. Once the asset pricing model under consideration includes more factors with greater non-normalities, the differences in terms of efficiency and robustness clearly emerge. In this sense, our results are closer to those on Kan and Zhou (1999), Kan and Zhou (2001), and Lozano and Rubio (2011).

The purpose of this work is to contribute to the understanding of the finite sample properties of the Beta and SDF methods by showing evidence about the elements which determine the parameters bias. Further extensions to this work could explore what happens when considering non-traded factors. In principle, there is no reason to expect a similar pattern. Other kinds of extensions would include the impact over other more specific areas in finance which traditionally follow one specific method.

References

Ahn, S. C. and C. Gadarowski (2004). Small sample properties of the model specification test based on the hansen-jagannathan distance. *Journal of Empirical Finance 11(1)*, 109–132.

Amsler, C. E. and P. Schmidt (1985). A monte carlo investigation of the accuracy of multivariate capm tests. *Journal of Financial Economics 14(3)*, 359–375.

Balduzzi, P. and T. Yao (2007). Testing heterogeneous-agent models: An alternative aggregation approach. *Journal of Monetary Economics 54(2)*, 369–412.

Balvers, R. J. and D. Huang (2007). Productivity-based asset pricing: Theory and evidence. *Journal of Financial Economics 86(2)*, 405–445.

Black, F., M. Jensen, and M. S. Scholes (1972). The capital asset pricing model: Some empirical tests. *Studies in Theory of Capital Markets. Jensen, Ed. New York: Preager Publishers. 17*, 79–121.

Brandt, M. W. and D. A. Chapman (2018, jan). Linear approximations and tests of conditional pricing models. *Review of Finance*.

Burnside, A. C. (2007, aug). Empirical asset pricing and statistical power in the presence of weak risk factors. Technical report, NBER.

Cai, Z. and Y. Hong (2009). Some recent developments in nonparametric finance. *Advances in Econometrics 25*, 379–432.

Campbell, J. Y., A. W. Lo, and A. C. MacKinlay (1997). *The Econometrics of Financial Markets*. Princeton NJ: Princeton University Press.

Carhart, M. M. (1997). On persistence in mutual fund performance. *Journal of Finance 52(1)*, 57–82.

Chen, R. and R. Kan (2004). Finite sample analysis of two-pass cross-sectional regressions. *The Review of Financial Studies (forthcoming).*

Cochrane, J. (2001, oct). A rehabilitation of stochastic discount factor methodology. Technical report, NBER.

Cochrane, J. H. (2005). *Asset Pricing - Revised Edition*. Princeton University Press.

Dittmar, R. F. (2002, feb). Nonlinear pricing kernels, kurtosis preference, and evidence from the cross section of equity returns. *The Journal of Finance 57*(1), 369–403.

Dybvig, P. H. and J. E. Ingersoll (1982). Mean variance theory in complete markets. *Journal of Business 55(2)*, 233–251.

Fama, E. F. and K. R. French (1992). The cross-section of expected stock returns. *Journal of Finance 47(2)*, 427–465.

Fama, E. F. and K. R. French (1993). Common risk factors in the returns on stocks and bonds. *Journal of Financial Economics 33(1)*, 3–56.

Fama, E. F. and J. D. MacBeth (1973). Risk return and equilibrium: empirical tests. *Journal of Financial Political Economy 81(3)*, 607–636.

Farnsworth, H., W. Ferson, D. Jackson, and S. Todd (2002). Performance evaluation with stochastic discount factors. *Journal of Business 75(3)*, 473–503.

Ferson, W. E. (2005). *Tests of Multifactor Pricing Models, Volatility Bounds and Portfolio Performance*. Handbook of the Economics of Finance. Elsevier Science Publishers, North Holland.

Ferson, W. E. and R. Jagannathan (1996). *Econometric Evaluation of Asset Pricing Models*. G. S. Maddala and C. R. Rao, Eds. Handbook of Statistics, Vol. 14: Statistical Methods in Finance. Elsevier.

Graham, J. R. and C. R. Harvey (2001, may). The theory and practice of corporate finance: evidence from the field. *Journal of Financial Economics 60*(2-3), 187–243.

Guidolin, M. and A. Timmermann (2008, April). International asset allocation under regime switching, skew, and kurtosis preferences. *Review of Financial Studies 21*(2), 889–935.

Hansen, L. P. (1982). Large sample properties of generalized method of moments estimators. *Econometrica 50(4)*, 1029–1054.

Hansen, L. P. and R. Jagannathan (1997). Assessing specification errors in stochastic discount factor models. *Journal of Finance 55(2)*, 557–590.

Hansen, L. P. and S. F. Richard (1987). The role of conditioning information in deducing testable. *Econometrica 55(3)*, 587–613.

Harvey, C. R. and A. Siddique (2000, 06). Conditional skewness in asset pricing tests. *The Journal of Finance 55*(3), 1263–1295.

Ingersoll, J. E. (1987). *Theory of Financial Decision Making.* Rowman and Littlefield Publishing, Inc.

Jagannathan, R., G. Skoulakis, and Z. Wang (2002). Generalized method of moments: applications in finance. *Journal of Business and Economics Statistics 20(4)*, 470–481.

Jagannathan, R., G. Skoulakis, and Z. Wang (2008). The analysis of the cross section of security returns. *Handbook of Financial Econometrics. Edited By Yacine Ait-Sahalia and Lars Hansen 1*, ch. 14.

Jagannathan, R. and Z. Wang (1996). The conditional capm and the cross-section of expected returns. *Journal of Finance 51(1)*, 3–53.

Jagannathan, R. and Z. Wang (1998). An asymptotic theory for estimating beta-pricing models using cross-sectional regression. *Journal of Finance 53(4)*, 1285–1309.

Jagannathan, R. and Z. Wang (2002). Empirical evaluation of asset-pricing models: A comparison of the sdf and beta methods. *Journal of Finance 57(5)*, 2337–2367.

Julliard, C. and J. A. Parker (2005). Consumption risk and the cross-section of expected returns. *Journal of Political Economy 113(1)*, 185–222.

Kan, R. and C. Robotti (2008). Specification tests of asset pricing models using excess returns. *Journal of Empirical Finance 15(5)*, 816–838.

Kan, R. and C. Robotti (2009). A note on the estimation of asset pricing models using simple regression betas. *SSRN Electronic Journal*.

Kan, R., C. Robotti, and J. Shanken (2013). Pricing model performance and the two-pass cross-sectional regression methodology. *The Journal of Finance 68*(6), 2617–2649.

Kan, R. and G. Zhou (1999). A critique of the stochastic discount factor methodology. *Journal of Finance 54(4)*, 1221–1248.

Kan, R. and G. Zhou (2001). Empirical asset pricing: the beta method versus the stochastic discount factor method. *Unpublished working paper. Rotman School of Management, University of Toronto*.

Kan, R. and G. Zhou (2017, feb). Modeling non-normality using multivariate t: implications for asset pricing. *China Finance Review International 7*(1), 2–32.

Lewellen, J., S. Nagel, and J. Shanken (2009). A skeptical appraisal of asset pricing tests. *Journal of Financial Economics Forthcoming (2009)*, 175–194.

Lintner, J. (1965). Security prices, risk and maximal gains from diversification. *Journal of Finance 20(4)*, 587–615.

Lozano, M. and G. Rubio (2011, jan). Evaluating alternative methods for testing asset pricing models with historical data. *Journal of Empirical Finance 18*(1), 136–146.

MacKinlay, A. C. and M. P. Richardson (1991). Using generalized method of moments to test mean-variance efficiency. *Journal of Finance 46(2)*, 511–527.

Mossin, J. (1966). Equilibrium in a capital asset market. *Econometrica 34(4)*, 768–783.

Nieto, B. and R. Rodríguez (2005). Modelos de valoración de activos condicionales: un panorama comparativo. *Investigaciones Económicas XXIX (1)*, 33–72.

Peñaranda, F. and E. Sentana (2015, may). A unifying approach to the empirical evaluation of asset pricing models. *Review of Economics and Statistics 97*(2), 412–435.

Shanken, J. and G. Zhou (2007). Estimating and testing beta pricing models: Alternative method and their performance in simulations. *Journal of Financial Economics 84(1)*, 40–86.

Sharpe, W. F. (1964). Capital asset prices: A theory of market equilibrium under conditions of risk. *Journal of Finance 19(3)*, 425–442.

Smith, P. and M. Wickens (2002). Asset pricing with observable stochastic discount factors. *Journal of Economic Surveys 16(3)*, 397–446.

Vassalou, M., Q. Li, and Y. Xing (2006). Sector investment growth rates and the cross-section of equity returns. *Journal of Business 79(3)*, 1637–1636.

Velu, R. and G. Zhou (1999). Testing multi-beta asset pricing models. *Journal of Empirical Finance 6(3)*, 219–241.

Wang, Z. and X. Zhang (2006). Empirical evaluation of asset pricing models: Arbitrage and pricing errors over contingent claims. Technical report, Staff Report, Federal Reserve Bank of New York.

Yogo, M. (2006). A consumption-based explanation of expected stock returns. *The Journal of Finance 61(2)*, 539–580.

Appendix A. Proof Proposition 1 – Higher-order Moments in Empirical Asset Pricing Estimation

Proof. Before calculating the asymptotic variance of the SDF and the Beta methods, we define tensor operations. Let $T1$ be a tensor of dimension $N_1 \times N_2 \times \cdots \times N_p$, and $T2$ a tensor of dimension $M_1 \times M_2 \times \cdots \times M_o$, with all the elements $N_1, \ldots, N_p, M_1, \ldots, M_o$ greater than one and $p > o$ without loss of generality, we define the **expansion tensor product**,

$$\otimes_E(T1, T2)_{i_1,\ldots,i_m,i_m+1,\ldots,i_{m+p}} = T1_{i_1,\ldots,i_p} \times T2_{i_1,\ldots,i_o}$$

as the result of the expansion of tensors $X1$ and $X2$ in a tensor of dimension $N_1 \times N_2 \times \cdots \times N_p \times M_1 \times M_2 \times \cdots \times M_o$. Consider the case where $N_1 = M_1, \ldots, N_o = M_o$. The **reduction tensor product**, is defined as,

$$\otimes_R(T1, T2)_{i_{p+1},\ldots,i_{p+o}} = T1_{i_1,\ldots,i_o} \times T2_{i_1,\ldots,i_o},$$

the tensor of reduced dimension $N_{p+1} \times N_{p+2} \times \cdots \times N_o$ that results from the dot-product of tensors $T1$ and $T2$.

A.1 Asymptotic variance SDF method

In the case of the SDF method, we calculate the GMM estimator asymptotic variance. First, we consider the general case where δ, and μ in equation (13) can be different. Define $g_s(r_t, f_t, \lambda) = r_t(1 - \lambda f_t)$, the covariance matrix of $g_s(r_t, f_t, \lambda)$, in the case the factor f is non-Gaussian and has higher-order moments, is:

$$\begin{aligned}
S_s &= E[g_s(r_t, f_t, \lambda) g_s(r_t, f_t, \lambda)'] \\
&= \mathbf{B}\Big(\otimes_R \left(\otimes_R \left(\kappa_4, \lambda\right), \lambda\right) + \\
&\qquad 2\delta \left(\otimes_R \left(\kappa_3, \lambda\right) \lambda\right)' + 2 \otimes_R \left(\kappa_3, \lambda\lambda'\mu - \lambda\right) + \\
&\qquad \mathbf{\Omega} \left(\mathbf{I}_N + 4(\lambda\lambda'\mu - \lambda)\delta' + (\lambda'\mu\mu'\lambda)\mathbf{I}_N - 2\lambda'\mu\mathbf{I}_N\right) + \delta\delta'\lambda'\mathbf{\Omega}\lambda + \\
&\qquad (\delta\delta' + \lambda'\mu\mu'\lambda\delta\delta' - 2\lambda'\mu\delta\delta')\Big)\mathbf{B}' + \\
&\quad (1 - 2\lambda'\mu + \lambda'\mu\mu'\lambda + \lambda'\mathbf{\Omega}\lambda)\, \mathbf{\Sigma}_{\epsilon_t}.
\end{aligned} \tag{A1}$$

The elements in (A1) are sorted from the more complex (a tensor of fourth-order, to most simple (a tensor of second order – a matrix). Higher-order moments inside (A1) are the result of higher-order expected values of the multivariate factor f_t. These elements will not appear in a single factor analysis such as Kan and Zhou (1999) or Jagannathan and Wang (2002). We split the elements of (A1). Define:

$$\begin{aligned}\mathbf{A}_B &= \otimes_R\left(\otimes_R\left(\kappa_4,\lambda\right),\lambda\right)+2\delta\left(\otimes_R\left(\kappa_3,\lambda\right)\lambda\right)'+2\otimes_R\left(\kappa_3,\lambda\lambda'\mu-\lambda\right)+\\&\quad \boldsymbol{\Omega}\left(\mathbf{I}_N+4(\lambda\lambda'\mu-\lambda)\delta'+(\lambda'\mu\mu'\lambda)\mathbf{I}_N-2\lambda'\mu\mathbf{I}_N\right)+\delta\delta'\lambda'\boldsymbol{\Omega}\lambda+\\&\quad \left(\delta\delta'+\lambda'\mu\mu'\lambda\delta\delta'-2\lambda'\mu\delta\delta'\right),\end{aligned}$$

and

$$a_{\epsilon_t}=1-2\lambda'\mu+\lambda'\mu\mu'\lambda+\lambda'\boldsymbol{\Omega}\lambda,$$

then the covariance of $g_s(r_t,f_t,\lambda)$ can be written as:

$$S_s=\mathbf{B}\mathbf{A}_B\mathbf{B}'+a_{\epsilon_t}\boldsymbol{\Sigma}_{\epsilon_t} \tag{A2}$$

The inverse of (A2) is:

$$S_s^{-1}=\frac{1}{a_{\epsilon_t}}\boldsymbol{\Sigma}_{\epsilon_t}^{-1}-\frac{1}{a_{\epsilon_t}^2}\boldsymbol{\Sigma}_{\epsilon_t}^{-1}\mathbf{B}\left(\mathbf{A}_B^{-1}+\frac{1}{a_{\epsilon_t}}\mathbf{B}'\boldsymbol{\Sigma}_{\epsilon_t}^{-1}\mathbf{B}\right)^{-1}\mathbf{B}'\boldsymbol{\Sigma}_{\epsilon_t}^{-1}. \tag{A3}$$

The partial derivatives of g_s respect to λ will produce a matrix:

$$D_s=E\left[\frac{\partial g_s}{\partial\lambda}\right]=-\mathbf{B}\left(\boldsymbol{\Omega}+\delta\mu'\right), \tag{A4}$$

Then, using (A3) and (A33), we have the asymptotic variance of the SDF model is:

$$Avar(\hat{\lambda})=\left(D_s'S_s^{-1}D_s\right)^{-1}. \tag{A5}$$

In the case of single factor models, and defining $\sigma^2=\Omega$, the variance of the single-factor, equations (A2), (A3), (A33), and (A5), have their equivalents in:

$$S_s = \frac{\sigma^2(\sigma^4+\delta^4)+\kappa_4\delta^2+2\kappa_3(\delta^3-\delta\sigma^2)-3\delta^2\sigma^4}{(\sigma^2+\mu\delta)^2}\beta\beta'+\frac{\sigma^2(\sigma^2+\delta^2)}{(\sigma^2+\mu\delta)^2}\boldsymbol{\Sigma}_{\epsilon_t}, \tag{A6}$$

$$\begin{aligned}S_s^{-1} &= \frac{(\sigma^2+\mu\delta)^2}{\sigma^2(\sigma^2+\delta^2)}\boldsymbol{\Sigma}_{\epsilon_t}^{-1}-\frac{(\sigma^2+\mu\delta)^2}{\sigma^2(\sigma^2+\delta^2)}\times\\&\quad\left(\beta'\boldsymbol{\Sigma}_{\epsilon_t}^{-1}\beta+\frac{\sigma^2(\sigma^4+\delta^4)+\kappa_4\delta^2+2\kappa_3(\delta^3-\delta\sigma^2)-3\delta^2\sigma^2}{\sigma^2(\sigma^2+\delta^2)}\right)^{-1}\boldsymbol{\Sigma}_{\epsilon_t}^{-1}\beta\beta'\boldsymbol{\Sigma}_{\epsilon_t}^{-1},\end{aligned} \tag{A7}$$

$$D_s = E\left[\frac{\partial g_s}{\partial\lambda}\right]=-(\sigma^2+\mu\delta)\beta, \tag{A8}$$

$$Avar(\hat{\lambda}) = \frac{\sigma^2(\sigma^2+\delta^2)}{(\sigma^2+\mu\delta)^4}\left(\beta'\boldsymbol{\Sigma}_{\epsilon_t}^{-1}\beta\right)^{-1}+\frac{\sigma^2(\sigma^4+\delta^4)}{(\sigma^2+\mu\delta)^4}+\frac{\kappa_4\delta^2+2\kappa_3(\delta^3-\delta\sigma^2)-3\delta^2\sigma^2}{(\sigma^2+\mu\delta)^4} \tag{A9}$$

The equivalent asymptotic variance (A9) in the single factor Gaussian case is (Jagannathan et al., 2002):

$$Avar(\hat{\lambda}) = \frac{\sigma^2(\sigma^2+\delta^2)}{(\sigma^2+\mu\delta)^4}\left(\beta'\mathbf{\Sigma}_{\epsilon_t}^{-1}\beta\right)^{-1} + \frac{\sigma^2(\sigma^4+\delta^4)}{(\sigma^2+\mu\delta)^4}. \tag{A10}$$

The difference in asymptotic variance of the SDF method when modelling a Gaussian factor, and a non-Gaussian vector comes from the higher-order moments terms:

$$\frac{\kappa_4\delta^2 + 2\kappa_3\delta(\delta^2-\sigma^2) - 3\delta^2\sigma^2}{\sigma^2(\sigma^2+\delta^2)}. \tag{A11}$$

The additional term (A11) will increase the asymptotic variance for heavy tailed distributions (greater κ_4), and will decrease by higher negative skewness (lower values of κ_3).

A.2 Asymptotic variance Beta method - joint estimation

We calculate the GMM asymptotic variance of the risk premiums for the case of multifactor Beta models. First, we solve for the general case where the parameters, $\theta = (\delta, \mathbf{B}, \mu, \sigma^2)$, are estimated jointly as in Jagannathan and Wang (2002). Define

$$\begin{aligned} g_b(r_t, f_t, \theta) = \begin{pmatrix} g_b(1) \\ g_b(2) \\ g_b(3) \\ g_b(4) \end{pmatrix} &= \begin{pmatrix} r_t - \mathbf{B}(\delta + f_t - \mu) \\ (r_t - \mathbf{B}(\delta + f_t - \mu))f_t' \\ f_t - \mu \\ (f_t-\mu)(f_t-\mu)' - \mathbf{\Omega} \end{pmatrix} \\ &= \begin{pmatrix} \epsilon_t \\ \epsilon_t f_t' \\ f_t - \mu \\ (f_t-\mu)(f_t-\mu)' - \mathbf{\Omega} \end{pmatrix}, \end{aligned} \tag{A12}$$

where $\theta = (\delta, \mathbf{B}, \mu, \mathbf{\Omega})$. The covariance of g_b (A12) is

$$S_b = \begin{pmatrix} \mathbf{\Sigma}_{\epsilon_t} & \otimes_E(\mathbf{\Sigma}_{\epsilon_t}, \mu) & 0 & 0 \\ \otimes_E(\mathbf{\Sigma}_{\epsilon_t}, \mu) & \otimes_E(\mathbf{\Sigma}_{\epsilon_t}, \mathbf{\Omega}) & 0 & 0 \\ 0 & 0 & \mathbf{\Omega} & \kappa_3 \\ 0 & 0 & \kappa_3 & \kappa_4 - \otimes_E(\mathbf{\Omega}, \mathbf{\Omega}) \end{pmatrix}. \tag{A13}$$

We need to calculate the partial derivatives of g_b respect to the parameters θ. The first partial derivative, $\frac{\partial g_b(1)}{\partial \delta} = \mathbf{B}$. The partial derivative $\frac{\partial g_b(2)}{\partial \delta}$ will produce the third-order tensor,

$$\frac{\partial g_b(2)}{\partial \delta} = -\otimes_E(\mathbf{B}, \mu).$$

The following partial derivatives are null: $\frac{\partial g_b(3)}{\partial \delta} = \frac{\partial g_b(4)}{\partial \delta} = \frac{\partial g_b(3)}{\partial \mathbf{B}} = \frac{\partial g_b(4)}{\partial \mathbf{B}} = \frac{\partial g_b(1)}{\partial \mu} = \frac{\partial g_b(2)}{\partial \mu} = \frac{\partial g_b(1)}{\partial \mathbf{\Omega}} = \frac{\partial g_b(2)}{\partial \mathbf{\Omega}} = 0$.

In the case of $\frac{\partial g_b(1)}{\partial \mathbf{B}}$, it will produce the following $N \times N \times K$ third-order tensor:

$$\frac{\partial g_b(1)}{\partial \mathbf{B}} = -\left\{ \begin{pmatrix} & \delta' & & \\ 0 & 0 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 0 \end{pmatrix}_{N\times K}, \dots, \begin{pmatrix} 0 & 0 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 0 \\ & \delta' & & \\ 0 & 0 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 & 0 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 0 \\ & \delta' & & \end{pmatrix} \right\}_{N\times N\times K}.$$

We need to define some tensor notations. Let us define the canonical basis:

$$\mathbf{e}' = [\mathbf{e}_{1,:}, \mathbf{e}_{2,:}, \dots, \mathbf{e}_{N,:}]_{1\times N}.$$

where every element of this vector is a matrix:

$$\mathbf{e}_{i,:} = \underbrace{\begin{pmatrix} 0 & 0 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 0 \\ 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ & \vdots & & \\ 0 & 0 & \dots & 0 \end{pmatrix}_{N\times K}}_{i-th \text{ row equal to one.}},$$

The identity tensor can be denoted using tensor notation as $\mathbf{I}_{N\times N\times K} = \otimes_E (\mathbf{e}, \mathbf{1}_{N\times 1})$. Then, we can denote $\frac{\partial g_b(1)}{\partial \mathbf{B}}$ as

$$\delta \mathbf{I}_{N\times N\times K} \equiv \otimes_E (\delta \mathbf{e}, \mathbf{1}_{N\times 1}), \tag{A14}$$

where $\delta\mathbf{e} = \left[[\delta, \dots, \delta]' \odot \mathbf{e}_{1,:}, \dots, [\delta, \dots, \delta]' \odot \mathbf{e}_{N,:}\right]$ and $\odot$ is the element wise multiplication. The partial derivative $\frac{\partial g_b(2)}{\partial \mathbf{B}}$ will produce a fourth-order tensor:

$$
\frac{\partial g_b(2)}{\partial \mathbf{B}} =
$$

$$
-\left\{\begin{array}{ccc}
\begin{pmatrix} \delta_1\mu_1+\mathbf{\Omega}_{1,1} & \dots & \delta_1\mu_k+\mathbf{\Omega}_{1,k} \\ 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \end{pmatrix}_{N\times K} & \dots & \begin{pmatrix} \delta_k\mu_1+\mathbf{\Omega}_{k,1} & \dots & \delta_k\mu_k+\mathbf{\Omega}_{k,k} \\ 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & \dots & 0 \\ \delta_1\mu_1+\mathbf{\Omega}_{1,1} & \dots & \delta_1\mu_k+\mathbf{\Omega}_{1,k} \\ 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \end{pmatrix}_{N\times K} & \dots & \begin{pmatrix} 0 & \dots & 0 \\ \delta_k\mu_1+\mathbf{\Omega}_{k,1} & \dots & \delta_k\mu_k+\mathbf{\Omega}_{k,k} \\ 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \end{pmatrix} \\
 & \vdots & \\
\begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \\ \delta_1\mu_1+\mathbf{\Omega}_{1,1} & \dots & \delta_1\mu_k+\mathbf{\Omega}_{1,k} \end{pmatrix}_{N\times K} & \dots & \begin{pmatrix} 0 & \dots & 0 \\ \vdots & & \\ 0 & \dots & 0 \\ \delta_k\mu_1+\mathbf{\Omega}_{k,1} & \dots & \delta_k\mu_k+\mathbf{\Omega}_{k,k} \end{pmatrix}
\end{array}\right\}_{N\times K\times N\times K} \tag{A15}
$$

Using a similar notation as in (A14), we denote the fourth-order identity tensor and the corresponding partial derivative $\frac{\partial g_b(2)}{\partial \mathbf{B}}$ as

$$
\mathbf{I}_{N\times N\times K\times K} = \otimes_E \left(\mathbf{e}, \mathbf{1}_{N\times K}\right), \tag{A16}
$$

$$
\frac{\partial g_b(2)}{\partial \mathbf{B}} = \left(\delta\mu' + \mathbf{\Omega}\right)\mathbf{I}_{N\times N\times K\times K} \equiv \otimes_E \left(\left(\delta\mu' + \mathbf{\Omega}\right)\mathbf{e}, \mathbf{1}_{N\times N}\right). \tag{A17}
$$

The partial derivatives of the respect to the mean and the variance of the factor are $\frac{\partial g_b(3)}{\partial \mu} = -\mathbf{I}_{K\times 1}$ and $\frac{\partial g_b(4)}{\partial \mu} = -\mathbf{I}_{K\times K}$. Then, the expected value of the partial derivatives $D_b = E\left[\frac{\partial g_b}{\partial \theta}\right]$ is the following matrix

$$
D_b = E\left[\frac{\partial g_b}{\partial \theta'}\right] = \begin{pmatrix} -\mathbf{B} & -\delta\mathbf{I}_{N\times N\times K} & \mathbf{B} & 0 \\ -\otimes_E\left(\mathbf{B}, \mu\right) & -\left(\delta\mu' + \mathbf{\Omega}\right)\mathbf{I}_{N\times N\times K\times K} & \otimes_E\left(\mathbf{B}, \mu\right) & 0 \\ 0 & 0 & -\mathbf{I}_{K\times 1} & 0 \\ 0 & 0 & 0 & -\mathbf{I}_{K\times K} \end{pmatrix}. \tag{A18}
$$

Define

$$
S_b^{-1} = \mathrm{inv}(S_b), \tag{A19}
$$

then, from the resulting matrix $V = \left(D_b' S_b^{-1} D_b\right)^{-1}$, the asymptotic variance of the δ^* parameter is equal to the top left corner element of the matrix:

$$Avar(\delta^*) = V_{1,1}.$$

Using the Delta method, and the definition of λ in (19), the asymptotic variance for the risk premium of the Beta method with multiple non-Gaussian factors is

$$\begin{aligned} Avar(\lambda^*) &= \left(\frac{\partial \lambda}{\partial \delta}\right)\left(\frac{\partial \lambda}{\partial \delta}\right)' Avar(\delta^*) \\ &= \left(-\delta\left(\left(\left(\mathbf{\Omega}+\delta\mu'\right)\left(\mathbf{\Omega}+\delta\mu'\right)'\right)^{-1}\mu\right)' + \mathbf{1}_{K\times K}\left(\mathbf{\Omega}+\delta\mu'\right)^{-1}\right) V_{1,1}. \end{aligned} \tag{A20}$$

The calculation of the asymptotic variance of the risk premium by using the Beta method, for the case a single non-Gaussian factor has the corresponding equations to the multifactor equivalents (A12), (A13), (A18), (A19), in

$$g_b(r_t, f_t, \theta) = \begin{pmatrix} r_t - (\delta + f_t - \mu)\beta \\ (r_t - (\delta + f_t - \mu)\beta) f_t \\ f_t - \mu \\ (f_t - \mu)^2 - \sigma^2 \end{pmatrix} = \begin{pmatrix} \epsilon_t \\ \epsilon_t f_t \\ f_t - \mu \\ (f_t - \mu)^2 - \sigma^2 \end{pmatrix}, \tag{A21}$$

$$S_b = \begin{pmatrix} \Omega & \mu\Omega & 0 & 0 \\ \mu\Omega & (\mu^2+\sigma^2)\Omega & 0 & 0 \\ 0 & 0 & \sigma^2 & \kappa_3 \\ 0 & 0 & \kappa_3 & \kappa_4 - \sigma^4 \end{pmatrix}, \tag{A22}$$

$$S_b^{-1} = \frac{1}{\sigma^2}\begin{pmatrix} (\mu^2+\sigma^2)\Omega^{-1} & -\mu\Omega^{-1} & 0 & 0 \\ -\mu\Omega^{-1} & \Omega^{-1} & 0 & 0 \\ 0 & 0 & -\dfrac{\sigma^2(\kappa_4-\sigma^4)}{\sigma^6-\kappa_4\sigma^2+\kappa_3^2} & \dfrac{\kappa_3\sigma^2}{\sigma^6-\kappa_4\sigma^2+\kappa_3^2} \\ 0 & 0 & \dfrac{\kappa_3\sigma^2}{\sigma^6-\kappa_4\sigma^2+\kappa_3^2} & \dfrac{\sigma^4}{\sigma^6-\kappa_4\sigma^2+\kappa_3^2} \end{pmatrix}, \tag{A23}$$

$$D_b = E\left[\frac{\partial g_b}{\partial \theta'}\right] = \begin{pmatrix} -\beta & -\delta I_n & \beta & 0 \\ -\mu\beta & -(\sigma^2+\mu\delta)I_n & \mu\beta & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{A24}$$

We calculate $\left(D_b' S_b^{-1} D_b\right)^{-1}$

$$\left(D_b' S_b^{-1} D_b\right)^{-1} = \begin{pmatrix} \frac{\sigma^2+\delta^2}{\sigma^2}\left(\beta'\Omega^{-1}\beta\right)^{-1}+\sigma^2 & -\frac{\delta}{\sigma^2}\left(\beta'\Omega^{-1}\beta\right)^{-1}\beta' & \sigma^2 & \kappa_3 \\ -\frac{\delta}{\sigma^2}\left(\beta'\Omega^{-1}\beta\right)^{-1}\beta & \frac{1}{\sigma^2+\delta^2}\Omega+\frac{\delta^2\left(\beta'\Omega^{-1}\beta\right)^{-1}\beta\beta'}{\sigma^2(\sigma^2+\delta^2)} & 0 & 0 \\ \sigma^2 & 0 & \sigma^2 & \kappa_3 \\ \kappa_3 & 0 & \kappa_3 & \kappa_4-\sigma^4 \end{pmatrix} \tag{A25}$$

The asymptotic variance of the GMM estimation of δ^* for the single factor non-Gaussian case is

$$Avar(\delta^*) = \frac{\sigma^2+\delta^2}{\sigma^2}\left(\beta'\mathbf{\Sigma}_{\epsilon_t}^{-1}\beta\right)^{-1}+\sigma^2. \tag{A26}$$

Applying the Delta method,[13] the corresponding asymptotic variance of the λ^* parameter –equivalent to (A20)– is[14,15]

$$Avar(\lambda^*) = \frac{\sigma^2(\sigma^2+\delta^2)}{(\sigma^2+\mu\delta)^4}\left(\beta'\mathbf{\Sigma}_{\epsilon_t}^{-1}\beta\right)^{-1}+\frac{\sigma^6}{(\sigma^2+\mu\delta)^4}. \tag{A27}$$

Considering (A5), (A20), and by applying some algebra with the support of equations (A10) and (A27) the asymptotic variance of the risk-premium estimator result is yield.

A.3 Asymptotic variance Beta method - separate estimation

In our paper, $\mu=\delta$, and the estimation of the parameter $\mathbf{B}$ is separate from the estimation of the parameters $\theta^*=(\delta,\sigma^2)=(\mu,\sigma^2)$. Then, we have that to estimate the asymptotic variance of the parameter $\delta^*=\mu^*$, we define

$$g_b(r_t,f_t,\theta) = \begin{pmatrix} f_t-\mu \\ (f_t-\mu)(f_t-\mu)'-\mathbf{\Omega} \end{pmatrix}, \tag{A28}$$

$$S_b = \begin{pmatrix} \mathbf{\Omega} & \kappa_3 \\ \kappa_3 & \kappa_4-\otimes_E(\mathbf{\Omega},\mathbf{\Omega}) \end{pmatrix}, \tag{A29}$$

[13]The use of the delta method requires that the parameter estimation–given the sequence X_t–converges to a normal distribution, $\sqrt{T}\,|X_T-\theta| \xrightarrow{D} N(0,\sigma^2)$. In the case the distribution of the factor deviates from the normal distribution, the estimated parameters might deviate from the normal, and the Delta approximation might underestimate the asymptotic variance. In our case, as we estimate δ separately from $\mathbf{B}$ in the next subsection, we use the GMM asymptotic results to provide an exact estimate of the asymptotic variance of δ without using the Delta method.

[14]The equation (A27) corrects the Jagannathan et al. (2002) approximation of the asymptotic variance by using the Beta method, that has a difference of $\frac{\sigma^2\delta^4}{(\sigma^2+\mu\delta)^4}$ between the Beta and the SDF methods.

[15]We can observe that in the non-Gaussian case, when using the Beta method, the higher-order moments do not affect the asymptotic variance of the risk premium estimation. This is consistent with the Beta method being a first- and second-order only asset pricing model. In the case of the SDF model, higher-order moments will discount risk premiums, therefore they will affect the asymptotic variance.

and

$$D_b = E\left[\frac{\partial g_b}{\partial \theta^{*\prime}}\right] = \begin{pmatrix} -\mathbf{I}_{K\times 1} & 0 \\ 0 & -\mathbf{I}_{K\times K} \end{pmatrix}, \tag{A30}$$

then

$$Avar(\lambda^*) = \left(-\delta\left(\left((\mathbf{\Omega}+\delta\mu')\left(\mathbf{\Omega}+\delta\mu'\right)'\right)^{-1}\mu\right)' + \mathbf{1}_{K\times K}\left(\mathbf{\Omega}+\delta\mu'\right)^{-1}\right)\mathbf{\Omega}. \tag{A31}$$

This is a Delta (first-order) approximation. A better approximation is made when considering the definition of λ into the GMM,

$$g_b(f_t, \lambda_b) = \mathrm{E}[r_t m_t] = \mathrm{E}[r_t(1-\lambda_b f_t)] = (r_t(1-\lambda_b f_t)) = 0,$$

That can be reduced to

$$g_b(f_t, \lambda_b) = (f_t(1-\lambda_b f_t)) = 0. \tag{A32}$$

The parameter λ_b is not estimated by the SDF method, but by the Beta method and using (19), $\lambda_b = \mu'\left(\mathbf{\Omega}+\mu\mu'\right)^{-1}$. The covariance matrix of $g_b(f_t,\theta)$ in (A32) is:

$$\begin{aligned} S_b &= \mathrm{E}[g_b(f_t,\lambda_b)g_b(f_t,\lambda_b)'] \\ &= \mathrm{E}[f_t f_t' - 2\otimes_R(\lambda_b, \otimes_E(f_t f_t', f_t)) + \otimes_R(\lambda_b\lambda_b', \otimes_E(f_t f_t', f_t f_t'))]. \end{aligned}$$

The partial derivatives of g_b respect to λ_b will produce a matrix

$$D_b = \mathrm{E}\left[\frac{\partial g_b}{\partial \lambda}\right] = -\left(\mathbf{\Omega}+\mu\mu'\right), \tag{A33}$$

Then, the asymptotic variance of λ^* is equal to

$$Avar(\lambda^*) = Avar(\lambda_b^*) = \left(\mathbf{\Omega}+\mu\mu'\right)^{-1} S_b\left(\left(\mathbf{\Omega}+\mu\mu'\right)^{-1}\right)'. \tag{A34}$$

In the case of single factors we have

$$Avar(\lambda^*) = \frac{((\sigma^2+\mu^2) - 2\lambda_b\mathrm{E}[f_t^3] + \lambda^2\mathrm{E}[f_t^4])}{(\sigma^2+\mu^2)}, \tag{A35}$$

with

$$\begin{aligned} \mathrm{E}[f_t^3] &= \kappa_3 + 3\sigma^2\mu + \mu^3, \\ \mathrm{E}[f_t^4] &= \kappa_4 + 4\kappa_3\mu + 3\sigma^4 + 6\sigma^2\mu^2 + \mu^4. \end{aligned}$$

A.4 Asymptotic variance pricing errors

We provide the asymptotic variances of the pricing errors. In both cases, SDF and Beta methods, the asymptotic variance of the pricing error is found by defining a sample mean of the estimator:

$$e_s(\hat{\lambda}) = \frac{1}{T}\left(\sum_{i=1}^{T} g_s(r_t, f_t, \lambda)\right), \tag{A36}$$

$$e_b(\theta) = \frac{1}{T}\left(\sum_{i=1}^{T} g_b(r_t, f_t, \theta)\right). \tag{A37}$$

The SDF pricing error $\hat{\pi}$ will be equal to (A36) (Jagannathan and Wang, 2002), then by Hansen (1982):

$$Avar\left(\hat{\pi}\right) = Avar\left(e_b(\hat{\lambda})\right) = \frac{1}{T}\left(\sum_{i=1}^{T} g_s(r_t, f_t, \lambda)\right) = S_s - D_s\left(D_s' S_s^{-1} D_s\right) D_s'. \tag{A38}$$

Consider that in the Beta method, the equivalent Jensen's α is:

$$\alpha^* = Q^* e(\theta^*) = \left[I_n, \mathbf{0}_{n\times n}, -\beta^*, \mathbf{0}_{n\times 1}\right] e(\theta^*). \tag{A39}$$

Using equations (A37), (A38) and (20) the asymptotic variance of the pricing error is yield:

$$Avar\left(\pi^*\right) = \left(\left(\mathbf{\Sigma}_{\epsilon_t} + \delta\mu'\right) \mathbf{\Sigma}_{\epsilon_t}^{-1} (\mathbf{\Sigma}_{\epsilon_t}^{-1})' \left(\mathbf{\Sigma}_{\epsilon_t} + \delta\mu'\right)\right) Q \left(S_b - D_b(D_b' S^{-1} D_b) D_b'\right) Q'. \tag{A40}$$

□

Appendix B. Tables

Table I. Descriptive statistics of factors and portfolios.

US factors and portfolios taken from Kenneth R. French library, January 1927 to December 2018 ($T = 1104$). UK factors and portfolios taken from Alan Gregory library, October 1980 to December 2017 ($T = 447$).

	Factors				Portfolios		
	Market	Size	Value	Momentum	10 size	25 size-value	30 industry
				US			
Mean	0.6471	0.2096	0.3682	0.6617	0.8642	0.8786	0.7239
Variance	28.58	10.22	12.17	22.02	46.8924	47.9440	31.3499
Skewness	0.19	1.93	2.18	-3.06	1.3791	1.3957	0.3583
Kurtosis	10.81	22.28	22.18	30.90	18.3248	18.2319	11.8010
				US - Recession Periods			
Mean	-0.4320	-0.1292	0.2696	0.6361	-0.4787	-0.4481	-0.3041
Variance	67.35	11.31	25.52	55.30	103.6166	108.4459	71.4597
Skewness	0.40	0.48	3.05	-3.24	1.4465	1.3667	0.5638
Kurtosis	6.83	5.02	22.84	21.98	12.3939	11.4992	7.5031
				UK			
Mean	0.5482	0.1318	0.2889	0.9794	0.8190	-	-
Variance	19.35	9.59	9.90	18.03	21.3262	-	-
Skewness	-1.01	-0.13	-0.51	-0.93	-0.7562	-	-
Kurtosis	6.86	5.82	9.41	8.50	6.6923	-	-

Table II. Parameter Values used in Monte Carlo Simulation: US data, 10 size-sorted portfolios.

The table presents the parameters used for the Monte Carlo simulation to estimate GMM under Beta and SDF methods. Parameters are estimated from single US factors and portfolios taken from Kenneth R. French library, January 1927 to December 2018 ($T = 1104$).

Panel A: Market Factor

μ =0.6471, σ =5.3457, δ =0.6471, λ =0.0223, κ_3 =28.4438, κ_4 =8830.8782

Decile Portfolios

Small	2	3	4	5	6	7	8	9	Large
				β					
1.4187	1.3854	1.3285	1.2572	1.2287	1.2037	1.1514	1.1143	1.0628	0.9311
				α					
0.1951	0.0750	0.1006	0.1094	0.0772	0.1037	0.0708	0.0675	0.0288	-0.0051
				$\mathbf{\Sigma}_\epsilon$					
40.9683	26.4672	19.2810	15.9265	11.5231	8.7891	6.9172	3.6288	1.7423	-2.9924
26.4672	21.0432	14.8231	12.5335	9.4669	7.1121	5.5034	3.0992	1.2590	-2.5093
19.2810	14.8231	12.7128	10.2017	7.7451	6.0527	4.5158	2.9101	1.2151	-2.1518
15.9265	12.5335	10.2017	9.9662	6.9964	5.6539	4.3061	2.7373	1.1963	-1.9928
11.5231	9.4669	7.7451	6.9964	6.3529	4.4908	3.4261	2.2858	1.0150	-1.6416
8.7891	7.1121	6.0527	5.6539	4.4908	4.5537	2.9280	2.1043	1.0706	-1.3824
6.9172	5.5034	4.5158	4.3061	3.4261	2.9280	3.2914	1.7930	0.9149	-1.1456
3.6288	3.0992	2.9101	2.7373	2.2858	2.1043	1.7930	2.1077	0.8522	-0.8831
1.7423	1.2590	1.2151	1.1963	1.0150	1.0706	0.9149	0.8522	1.2128	-0.4942
-2.9924	-2.5093	-2.1518	-1.9928	-1.6416	-1.3824	-1.1456	-0.8831	-0.4942	0.6792

Panel B: Size Factor

μ =0.2096, σ =3.1976, δ =0.2096, λ =0.0204, κ_3 =63.0820, κ_4 =2329.3833

Decile Portfolios

Small	2	3	4	5	6	7	8	9	Large
				β					
2.1964	1.9059	1.6321	1.4843	1.3097	1.1070	0.9908	0.8134	0.6319	0.3032
				α					
0.6529	0.5722	0.6183	0.6119	0.5979	0.6507	0.6082	0.6182	0.5841	0.5339
				$\mathbf{\Sigma}_\epsilon$					
49.1522	39.8210	36.4736	33.5470	31.9046	32.7070	31.3220	30.5135	30.6135	27.9171
39.8210	38.7365	35.5968	33.3644	32.5677	33.1717	31.7548	31.3391	30.9958	28.4157
36.4736	35.5968	35.8935	33.1421	32.5133	33.2535	31.6684	31.6150	30.9928	28.1090
33.5470	33.3644	33.1421	32.5886	31.2411	32.0761	30.6112	30.4023	29.7651	26.8311
31.9046	32.5677	32.5133	31.2411	31.9316	31.9058	30.5595	30.4927	29.8437	26.9634
32.7070	33.1717	33.2535	32.0761	31.9058	33.4033	31.2926	31.2015	30.4507	27.1876
31.3220	31.7548	31.6684	30.6112	30.5595	31.2926	31.1114	30.1903	29.4565	26.3927
30.5135	31.3391	31.6150	30.4023	30.4927	31.2015	30.1903	30.7998	29.4141	26.2191
30.6135	30.9958	30.9928	29.7651	29.8437	30.4507	29.4565	29.4141	29.3838	25.8014
27.9171	28.4157	28.1090	26.8311	26.9634	27.1876	26.3927	26.2191	25.8014	24.4915

Table III. Parameter Values used in Monte Carlo Simulation: US data, 10 size-sorted portfolios.

The table presents the parameters used for the Monte Carlo simulation to estimate GMM under Beta and SDF methods. Parameters are estimated from single US factors and portfolios taken from Kenneth R. French library, January 1927 to December 2018 ($T = 1104$).

Panel C: Value Factor									
μ =0.3682, σ =3.4880, δ =0.3682, λ =0.0299, κ_3 =92.6417, κ_4 =3282.2015									
Decile Portfolios									
Small	2	3	4	5	6	7	8	9	Large
β									
1.3215	1.0101	0.8781	0.7841	0.6631	0.6651	0.5611	0.5238	0.4976	0.2951
α									
0.6266	0.5996	0.6370	0.6343	0.6282	0.6377	0.6093	0.5958	0.5333	0.4888
$\mathbf{\Sigma}_\epsilon$									
77.2036	66.3564	58.9870	54.2554	50.6370	46.8608	44.5395	40.3482	36.7966	29.9806
66.3564	63.4395	56.5904	52.6359	49.9237	46.5579	44.1559	40.7427	37.1873	30.6963
58.9870	56.5904	53.7316	49.5204	47.2705	44.6110	42.1987	39.5844	36.2158	30.0150
54.2554	52.6359	49.5204	47.6225	44.7794	42.5227	40.2872	37.7428	34.6031	28.6167
50.6370	49.9237	47.2705	44.7794	44.1080	41.3551	39.2925	37.1520	34.2859	28.6418
46.8608	46.5579	44.6110	42.5227	41.3551	40.5448	37.9611	36.1645	33.5728	28.2310
44.5395	44.1559	42.1987	40.2872	39.2925	37.9611	37.3131	34.8500	32.4578	27.4493
40.3482	40.7427	39.5844	37.7428	37.1520	36.1645	34.8500	34.2225	31.4955	26.8597
36.7966	37.1873	36.2158	34.6031	34.2859	33.5728	32.4578	31.4955	30.4520	25.9737
29.9806	30.6963	30.0150	28.6167	28.6418	28.2310	27.4493	26.8597	25.9737	24.3724

Panel D: Momentum Factor									
μ =0.6617, σ =4.6930, δ =0.6617, λ =0.0295, κ_3 =-315.7859, κ_4 =14986.8391									
Decile Portfolios									
Small	2	3	4	5	6	7	8	9	Large
β									
-0.8723	-0.7138	-0.6682	-0.5838	-0.5502	-0.5373	-0.4702	-0.4723	-0.4548	-0.3434
α									
1.6904	1.4439	1.4025	1.3093	1.2364	1.2382	1.1270	1.1012	1.0175	0.8247
$\mathbf{\Sigma}_\epsilon$									
81.6857	68.8806	60.2655	55.6434	50.7276	47.2306	44.5265	39.6965	36.0591	28.1299
68.8806	64.6311	56.8770	53.0932	49.4241	46.2851	43.6598	39.7564	36.1533	28.9266
60.2655	56.8770	53.2794	49.3051	46.2587	43.8098	41.2739	38.2313	34.8395	28.1163
55.6434	53.0932	49.3051	47.5951	44.0312	41.9589	39.5941	36.6677	33.5025	27.0181
50.7276	49.4241	46.2587	44.0312	42.7920	40.2110	38.1225	35.6563	32.7902	26.8634
47.2306	46.2851	43.8098	41.9589	40.2110	39.5691	36.9378	34.8153	32.2180	26.5570
44.5265	43.6598	41.2739	39.5941	38.1225	36.9378	36.2747	33.5359	31.1455	25.9092
39.6965	39.7564	38.2313	36.6677	35.6563	34.8153	33.5359	32.6494	29.9370	25.1703
36.0591	36.1533	34.8395	33.5025	32.7902	32.2180	31.1455	29.9370	28.9096	24.3221
28.1299	28.9266	28.1163	27.0181	26.8634	26.5570	25.9092	25.1703	24.3221	22.8367

Table IV. Expected value and standard errors for CAPM model: US data, 10 size-sorted portfolios.

The table presents the expected value and the standard error of λ GMM estimates under the Beta and the SDF methods. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. The first estimator decorated with * are from the Beta method; the second and third correspond to the first and second-stage un-centred SDF method; and the fourth and fifth to the first and second-stage centred SDF method. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations.

T	$\mathrm{E}[\widehat{\lambda}^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\widehat{\lambda}^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
60	2.38	2.40	3.19	2.59	3.00	2.67	2.92	3.60	3.26	3.57
360	2.17	2.17	2.29	2.22	2.28	1.03	1.10	1.09	1.16	1.12
600	2.18	2.18	2.25	2.22	2.25	0.78	0.83	0.81	0.87	0.83
1000	2.14	2.14	2.18	2.18	2.20	0.62	0.66	0.63	0.68	0.65

Table V. Relative standard errors for CAPM model: US data, 10 size-sorted portfolios.

The table presents the relative standard errors of λ GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda}) \div \mathrm{E}(\widehat{\lambda})$. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. Estimators decorated with * are from the Beta method; with A and B to the un-centred and centred SDF method; and with 1 and 2 to the first and second-stage respectively. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations.

T	$\sigma_r(\widehat{\lambda}_1^U) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_2^U) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_1^C) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_2^C) \div \sigma_r(\widehat{\lambda}^*)$
60	1.0833	1.0045	1.1219	1.0581
360	1.0654	1.0046	1.0935	1.0324
600	1.0656	1.0030	1.0921	1.0288
1000	1.0658	1.0039	1.0918	1.0295

Table VI. Relative standard errors for Fama-French model: US data, 10 size-sorted portfolios.

The table presents the relative standard errors of λ GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda}) \div \mathrm{E}(\widehat{\lambda})$. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. Estimators decorated with * are from the Beta method; with A and B to the un-centred and centred SDF method; and with 1 and 2 to the first and second-stage respectively. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations.

T	$\sigma_r(\widehat{\lambda}_1^U) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_2^U) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_1^C) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_2^C) \div \sigma_r(\widehat{\lambda}^*)$
		Market		
60	0.9853	0.9649	1.0102	1.0042
360	1.0158	1.0024	1.0379	1.0247
600	1.0187	1.0043	1.0412	1.0279
1000	1.0229	1.0074	1.0447	1.0304
		Size		
60	0.7075	0.6684	0.5861	0.5613
360	1.1403	1.0304	1.1123	0.9963
600	1.1606	1.0453	1.1425	1.0230
1000	1.1717	1.0562	1.1600	1.0430
		Value		
60	2.3382	1.8642	2.7692	2.8484
360	3.0934	2.3801	3.1525	2.5604
600	3.1171	2.4012	3.1628	2.5191
1000	3.2271	2.4994	3.2641	2.5791

Table VII. Relative standard errors for RUH model: US data, 10 size-sorted portfolios.

The table presents the relative standard errors of λ GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda}) \div \mathrm{E}(\widehat{\lambda})$. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. Estimators decorated with * are from the Beta method; with A and B to the un-centred and centred SDF method; and with 1 and 2 to the first and second-stage respectively. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations.

T	$\sigma_r(\widehat{\lambda}_1^U) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_2^U) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_1^C) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_2^C) \div \sigma_r(\widehat{\lambda}^*)$
		Market		
60	0.9533	0.9134	0.9434	0.9232
360	1.0004	0.9868	1.0101	0.9992
600	0.9940	0.9842	1.0090	0.9991
1000	0.9924	0.9817	1.0109	0.9999
		Momentum		
60	2.4843	1.8939	20.7738	10.5442
360	3.3341	2.6710	5.2603	4.1297
600	3.4321	2.8270	4.5009	3.6950
1000	3.5491	2.8415	4.1864	3.3457
		Value		
60	1.6279	1.2432	2.5987	3.3452
360	2.6050	2.1012	3.0128	2.6525
600	2.7729	2.2583	3.0513	2.6226
1000	2.8326	2.2917	3.0234	2.5272

Table VIII. Relative standard errors for Carhart model: US data, 10 size-sorted portfolios.

The table presents the relative standard errors of λ GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda}) \div \mathrm{E}(\widehat{\lambda})$. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. Estimators decorated with * are from the Beta method; with A and B to the un-centred and centred SDF method; and with 1 and 2 to the first and second-stage respectively. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations.

T	$\sigma_r(\widehat{\lambda}_1^U) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_2^U) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_1^C) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_2^C) \div \sigma_r(\widehat{\lambda}^*)$
		Market		
60	0.9247	0.9078	0.9183	0.9261
360	0.9844	0.9787	0.9911	0.9861
600	0.9720	0.9658	0.9824	0.9775
1000	0.9737	0.9674	0.9877	0.9820
		Size		
60	0.0710	0.0682	0.0525	0.0525
360	0.7827	0.7292	0.6856	0.6543
600	0.8637	0.7912	0.7847	0.7320
1000	0.8916	0.8195	0.8352	0.7775
		Value		
60	1.5553	1.3347	4.1452	3.8806
360	3.0528	2.5920	4.1765	3.4226
600	3.1199	2.6862	3.7705	3.1938
1000	3.3112	2.8430	3.7456	3.1735
		Momentum		
60	2.4387	2.0441	17.4520	12.3867
360	3.1899	2.8505	5.1730	4.3170
600	3.1007	2.8569	4.1378	3.6745
1000	3.1815	2.9759	3.7893	3.4642

Table IX. Expected value and standard errors for Fama-French model: US data, 10 size-sorted portfolios.

The table presents the expected value and the standard error of λ GMM estimates under the Beta and the SDF methods. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. The first estimator decorated with * are from the Beta method; the second and third correspond to the first and second-stage un-centred SDF method; and the fourth and fifth to the first and second-stage centred SDF method. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations.

T	$\mathrm{E}[\widehat{\lambda}^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\widehat{\lambda}^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
					Market					
60	2.26	2.39	2.93	2.67	2.95	2.79	2.91	3.48	3.33	3.66
360	2.07	2.14	2.22	2.23	2.27	1.01	1.06	1.09	1.13	1.14
600	2.06	2.12	2.18	2.20	2.23	0.77	0.81	0.82	0.86	0.86
1000	2.06	2.13	2.16	2.19	2.21	0.59	0.62	0.62	0.66	0.65
					Size					
60	0.86	1.69	1.94	2.16	2.34	4.29	5.95	6.46	6.30	6.54
360	1.40	1.76	1.83	1.87	1.93	1.59	2.29	2.14	2.36	2.19
600	1.44	1.78	1.82	1.86	1.90	1.24	1.79	1.65	1.84	1.69
1000	1.48	1.81	1.84	1.88	1.91	0.93	1.34	1.23	1.38	1.26
					Value					
60	2.18	2.79	3.14	2.49	2.13	3.94	11.77	10.58	12.46	10.97
360	2.57	2.89	2.95	2.95	2.83	1.38	4.80	3.78	5.01	3.90
600	2.63	2.90	2.93	2.98	2.89	1.07	3.66	2.85	3.81	2.95
1000	2.64	2.86	2.87	2.94	2.88	0.81	2.81	2.19	2.92	2.26

Table X. Expected value and standard errors for RUH model: US data, 10 size-sorted portfolios.

The table presents the expected value and the standard error of λ GMM estimates under the Beta and the SDF methods. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. The first estimator decorated with * are from the Beta method; the second and third correspond to the first and second-stage un-centred SDF method; and the fourth and fifth to the first and second-stage centred SDF method. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations.

T	$E[\widehat{\lambda}^*]$	$E[\widehat{\lambda}_1^U]$	$E[\widehat{\lambda}_2^U]$	$E[\widehat{\lambda}_1^C]$	$E[\widehat{\lambda}_2^C]$	$\sigma(\widehat{\lambda}^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
					Market					
60	1.94	2.33	2.74	2.67	2.85	2.71	3.10	3.50	3.52	3.69
360	1.95	2.13	2.20	2.26	2.28	0.98	1.07	1.09	1.15	1.15
600	1.94	2.10	2.14	2.21	2.22	0.76	0.82	0.83	0.88	0.87
1000	1.95	2.10	2.13	2.20	2.21	0.59	0.63	0.63	0.67	0.67
					Momentum					
60	3.73	4.32	5.23	0.55	0.98	3.86	11.12	10.26	11.77	10.70
360	2.77	2.98	3.25	1.99	2.14	1.62	5.80	5.08	6.12	5.16
600	2.59	2.80	2.92	2.26	2.31	1.23	4.56	3.92	4.84	4.05
1000	2.52	2.69	2.80	2.41	2.48	0.94	3.57	2.97	3.77	3.09
					Value					
60	1.51	2.80	3.03	1.90	1.19	4.00	12.10	10.01	13.14	10.56
360	2.26	2.83	2.89	2.60	2.41	1.40	4.56	3.75	4.85	3.95
600	2.35	2.81	2.85	2.72	2.60	1.06	3.51	2.90	3.74	3.07
1000	2.39	2.84	2.87	2.83	2.76	0.80	2.70	2.20	2.87	2.34

Table XI. Expected value and standard errors for Carhart model: US data, 10 size-sorted portfolios.

The table presents the expected value and the standard error of λ GMM estimates under the Beta and the SDF methods. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. The first estimator decorated with * are from the Beta method; the second and third correspond to the first and second-stage un-centred SDF method; and the fourth and fifth to the first and second-stage centred SDF method. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations.

T	$\mathrm{E}[\widehat{\lambda}^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\widehat{\lambda}^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
					Market					
60	1.99	2.36	2.69	2.75	2.88	2.89	3.16	3.53	3.66	3.87
360	1.90	2.10	2.16	2.24	2.27	0.99	1.08	1.10	1.16	1.16
600	1.89	2.08	2.11	2.20	2.21	0.76	0.82	0.82	0.87	0.87
1000	1.90	2.08	2.10	2.19	2.20	0.59	0.63	0.64	0.68	0.67
					Size					
60	-0.08	1.64	1.80	2.43	2.48	4.47	6.41	6.75	7.01	7.15
360	0.90	1.76	1.79	2.08	2.06	1.61	2.46	2.33	2.55	2.41
600	0.99	1.73	1.77	1.97	1.97	1.25	1.89	1.77	1.96	1.82
1000	1.04	1.77	1.78	1.95	1.94	0.96	1.45	1.34	1.50	1.38
					Value					
60	1.40	2.72	2.96	1.12	1.10	4.13	12.47	11.64	13.65	12.64
360	2.17	2.76	2.85	2.14	2.29	1.43	5.55	4.87	5.89	5.16
600	2.26	2.89	2.90	2.53	2.59	1.09	4.33	3.75	4.59	3.98
1000	2.29	2.81	2.84	2.63	2.69	0.81	3.30	2.86	3.49	3.03
					Momentum					
60	3.83	4.25	4.98	0.65	0.89	4.07	11.01	10.81	11.98	11.69
360	2.69	2.89	3.11	1.89	2.13	1.62	5.54	5.33	5.86	5.53
600	2.57	2.79	2.90	2.21	2.35	1.24	4.17	4.01	4.41	4.17
1000	2.50	2.68	2.74	2.37	2.46	0.95	3.25	3.11	3.42	3.25

Table XII. Relative standard errors for four alternative single-factor models: US data, 10 size-sorted portfolios.

The table presents the relative standard errors of λ GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\lambda}) = \sigma(\widehat{\lambda}) \div \mathrm{E}(\widehat{\lambda})$. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. Estimators decorated with * are from the Beta method; with A and B to the un-centred and centred SDF method; and with 1 and 2 to the first and second-stage respectively. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations.

T	$\sigma_r(\widehat{\lambda}_1^U) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_2^U) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_1^C) \div \sigma_r(\widehat{\lambda}^*)$	$\sigma_r(\widehat{\lambda}_2^C) \div \sigma_r(\widehat{\lambda}^*)$
Single-factor model loaded with market factor				
60	1.0833	1.0045	1.1219	1.0581
360	1.0654	1.0046	1.0935	1.0324
600	1.0656	1.0030	1.0921	1.0288
1000	1.0658	1.0039	1.0918	1.0295
Single-factor model loaded with size factor				
60	1.8006	1.1771	1.7656	1.2153
360	1.7301	1.0869	1.7295	1.0965
600	1.6187	1.0576	1.6223	1.0660
1000	1.6199	1.0625	1.6249	1.0704
Single-factor model loaded with value factor				
60	3.6666	1.9399	3.4126	2.9384
360	2.9342	1.9542	2.9205	2.0909
600	2.9140	1.9689	2.9173	2.0602
1000	2.8233	1.9448	2.8397	2.0065
Single-factor model loaded with momentum factor				
60	3.0995	1.5323	3.0467	2.6295
360	1.8419	1.5265	1.8913	1.6745
600	1.8082	1.5198	1.8560	1.6095
1000	1.7837	1.5159	1.8224	1.5766

Table XIII. Expected value and standard errors for four alternative single-factor models: US data, 10 size-sorted portfolios.

The table presents the expected value and the standard error of λ GMM estimates under the Beta and the SDF methods. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. The first estimator decorated with * are from the Beta method; the second and third correspond to the first and second-stage un-centred SDF method; and the fourth and fifth to the first and second-stage centred SDF method. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations.

T	$\mathrm{E}[\widehat{\lambda}^*]$	$\mathrm{E}[\widehat{\lambda}_1^U]$	$\mathrm{E}[\widehat{\lambda}_2^U]$	$\mathrm{E}[\widehat{\lambda}_1^C]$	$\mathrm{E}[\widehat{\lambda}_2^C]$	$\sigma(\widehat{\lambda}^*)$	$\sigma(\widehat{\lambda}_1^U)$	$\sigma(\widehat{\lambda}_2^U)$	$\sigma(\widehat{\lambda}_1^C)$	$\sigma(\widehat{\lambda}_2^C)$
Single-factor model loaded with market factor										
60	2.38	2.40	3.19	2.59	3.00	2.67	2.92	3.60	3.26	3.57
360	2.17	2.17	2.29	2.22	2.28	1.03	1.10	1.09	1.16	1.12
600	2.18	2.18	2.25	2.22	2.25	0.78	0.83	0.81	0.87	0.83
1000	2.14	2.14	2.18	2.18	2.20	0.62	0.66	0.63	0.68	0.65
Single-factor model loaded with size factor										
60	1.66	1.69	2.02	1.82	1.84	4.31	7.90	6.19	8.34	5.81
360	1.80	1.77	1.87	1.79	1.85	1.64	2.78	1.85	2.82	1.84
600	1.83	1.86	1.90	1.88	1.88	1.26	2.06	1.37	2.09	1.38
1000	1.85	1.87	1.88	1.89	1.88	0.97	1.60	1.05	1.61	1.06
Single-factor model loaded with value factor										
60	2.84	2.79	3.33	3.22	2.09	3.94	14.17	8.97	15.24	8.51
360	2.93	2.90	3.05	2.99	2.87	1.44	4.19	2.93	4.30	2.95
600	2.92	2.91	2.97	2.98	2.88	1.09	3.16	2.18	3.24	2.21
1000	2.91	2.96	2.98	3.01	2.94	0.84	2.41	1.67	2.46	1.70
Single-factor model loaded with momentum factor										
60	4.30	4.39	5.79	5.51	3.25	3.96	12.52	8.15	15.44	7.85
360	3.01	3.06	3.29	3.22	3.00	1.75	3.26	2.91	3.53	2.91
600	2.89	2.91	3.05	3.02	2.89	1.35	2.45	2.16	2.61	2.17
1000	2.77	2.76	2.87	2.84	2.80	1.02	1.82	1.61	1.91	1.63

Table XIV. Relative standard errors for four alternative asset pricing models: US data, 10 size-sorted portfolios.

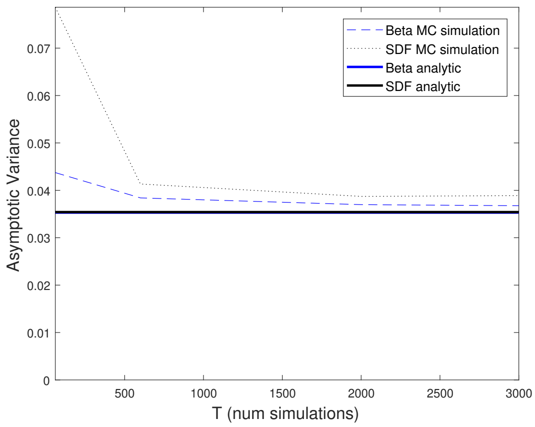
The table presents the relative standard errors of π GMM estimates under the Beta and the SDF methods, computed as $\sigma_r(\widehat{\pi}) = \sigma(\widehat{\pi}) \div \mathrm{E}(\widehat{\pi})$. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. Estimators decorated with * are from the Beta method; with A and B to the un-centred and centred SDF method; and with 1 and 2 to the first and second-stage respectively. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations.

T	$\sigma_r(\widehat{\pi}_1^U) \div \sigma_r(\widehat{\pi}^*)$	$\sigma_r(\widehat{\pi}_2^U) \div \sigma_r(\widehat{\pi}^*)$	$\sigma_r(\widehat{\pi}_1^C) \div \sigma_r(\widehat{\pi}^*)$	$\sigma_r(\widehat{\pi}_2^C) \div \sigma_r(\widehat{\pi}^*)$
		CAPM		
60	0.8246	1.1259	0.8250	1.0426
360	0.8338	1.0015	0.8340	0.9947
600	0.8237	1.0004	0.8238	0.9958
1000	0.8284	0.9996	0.8286	0.9987
		Fama-French		
60	0.6249	1.3654	0.6342	1.2793
360	0.6738	0.9997	0.6741	0.9041
600	0.6891	0.9475	0.6898	0.8846
1000	0.7220	0.9473	0.7226	0.9124
		RUH		
60	0.6245	1.2626	0.6274	1.2145
360	0.6003	1.0224	0.6014	0.9947
600	0.5896	0.9544	0.5897	0.9368
1000	0.6158	0.9572	0.6160	0.9352
		Carhart		
60	0.5807	1.2146	0.5823	1.1697
360	0.6667	1.0224	0.6650	0.9442
600	0.7317	0.9992	0.7303	0.9458
1000	0.8545	1.1034	0.8540	1.0734

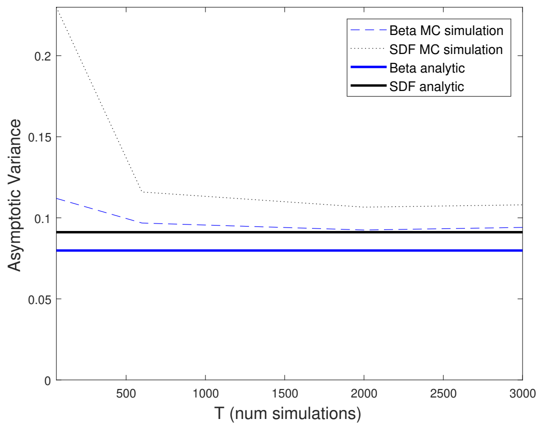
Table XV. Expected value and standard errors for four alternative asset pricing models: US data, 10 size-sorted portfolios.

The table presents the expected value and the standard error of π GMM estimates under the Beta and the SDF methods. The returns and factors are generated under the null hypothesis with the factors sampled from the empirical distribution. The first estimator decorated with * are from the Beta method; the second and third correspond to the first and second-stage un-centred SDF method; and the fourth and fifth to the first and second-stage centred SDF method. The results are presented for different lengths of time series observations, and they are based on 10,000 simulations.

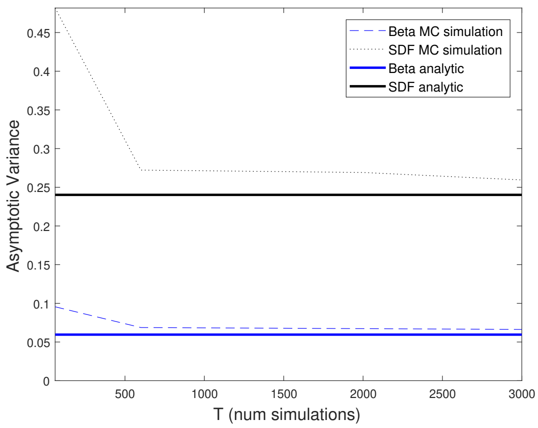
T	$\mathrm{E}[\widehat{\pi}^*]$	$\mathrm{E}[\widehat{\pi}_1^U]$	$\mathrm{E}[\widehat{\pi}_2^U]$	$\mathrm{E}[\widehat{\pi}_1^C]$	$\mathrm{E}[\widehat{\pi}_2^C]$	$\sigma(\widehat{\pi}^*)$	$\sigma(\widehat{\pi}_1^U)$	$\sigma(\widehat{\pi}_2^U)$	$\sigma(\widehat{\pi}_1^C)$	$\sigma(\widehat{\pi}_2^C)$
					CAPM					
60	35.39	23.50	47.11	24.27	41.30	20.05	10.98	30.04	11.34	24.39
360	14.63	9.74	15.13	9.90	15.01	8.23	4.57	8.52	4.64	8.40
600	11.25	7.50	11.48	7.61	11.48	6.35	3.48	6.48	3.54	6.45
1000	8.80	5.86	8.90	5.94	8.96	5.00	2.76	5.06	2.80	5.08
					Fama-French					
60	24.10	11.50	30.48	12.41	27.23	12.37	3.69	21.36	4.04	17.88
360	10.12	4.61	7.26	4.77	6.92	4.54	1.39	3.25	1.44	2.80
600	8.37	3.57	5.25	3.69	5.11	3.61	1.06	2.14	1.10	1.95
1000	7.07	2.76	3.85	2.84	3.81	2.89	0.82	1.49	0.84	1.42
					RUH					
60	36.68	12.38	36.33	13.49	35.80	20.01	4.22	25.02	4.61	23.71
360	15.46	5.02	10.59	5.27	10.66	8.28	1.62	5.80	1.70	5.68
600	12.33	3.89	7.79	4.07	7.93	6.65	1.24	4.01	1.29	4.00
1000	10.15	3.03	5.87	3.16	6.02	5.31	0.98	2.94	1.02	2.95
					Carhart					
60	32.97	9.99	27.49	11.08	26.98	19.16	3.37	19.41	3.75	18.34
360	16.28	4.17	6.87	4.39	6.78	7.98	1.36	3.44	1.43	3.14
600	14.49	3.24	4.89	3.40	4.91	6.31	1.03	2.13	1.08	2.02
1000	13.28	2.52	3.59	2.63	3.65	4.99	0.81	1.49	0.84	1.47



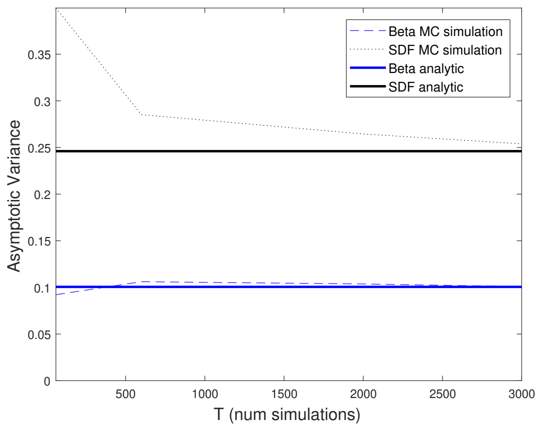
(a) Market risk factor



(b) Size factor



(c) Value factor



(d) Momentum factor

Figure 1. Asymptotic variance of the analytic and empirical estimated GMM with Beta and SDF methods, from a Monte Carlo simulation with data calibrated to the empirical observed market risk, size, value, and momentum factors from January 1927 to December 2018. Data from Kenneth R. French library.