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On Quadratic Forms in Multivariate Generalized Hyperbolic Random Vectors*

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Abstract

Countless test statistics can be written as quadratic forms in certain random vectors, or ratios thereof. Consequently, their distribution has received considerable attention in the literature. Except for a few special cases, no closed-form expression for the cdf exists, and one resorts to numerical methods. Traditionally the problem is analyzed under the assumption of joint Gaussianity; the algorithm that is usually employed is that of Imhof (1961). The present manuscript generalizes this result to the case of multivariate generalized hyperbolic random vectors. This flexible distribution nests, among others, the multivariate t, Laplace, and variance gamma distributions. An expression for the first partial moment is also obtained, which plays a vital role in financial risk management. The proof involves a generalization of the classic inversion formula due to Gil-Pelaez (1951). Two numerical applications are considered: first, the finite-sample distribution of the two stage least squares estimator of a structural parameter. Second, the Value at Risk and expected shortfall of a quadratic portfolio with heavy-tailed risk factors. An empirical application is examined, in which a portfolio of Dow Jones Industrial Index stock options is optimized with respect to its expected shortfall. The results demonstrate the benefits of the analytical expression.

Keywords: Characteristic Function; Conditional Value at Risk; Expected Shortfall; Transform Inversion; Two Stage Least Squares.

1 Introduction

The generalized hyperbolic distribution was introduced by Barndorff-Nielsen (1977) in the context of describing the log size of particles, but has since found applications in a variety of fields, including finance (Eberlein & Keller, 1995). Its multivariate extension has first been discussed in Blæsild & Jensen

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(1981). It, too, has received considerable attention in the literature; see, e.g., Hellmich & Kassberger (2011) and the references therein. The reason for its popularity is its flexibility: it can model thin, heavy, and semi-heavy tails, and nests many common distributions as special cases, such as the Student's t, Laplace, Variance Gamma, Hyperbolic, and Normal Inverse Gaussian distributions and their multivariate extensions.

The present manuscript is concerned with quadratic forms in multivariate generalized hyperbolic random vectors, or more precisely, the sum of a quadratic and a linear form. Quadratic forms arise in a variety of applications; for example, many testing problems in linear models lead to statistics whose null distribution can be expressed in this form. Some well-known examples are Durbin & Watson's (1950) test for autocorrelation, the "coefficient" form of Dickey & Fuller's (1979) test for a unit root, and the stationarity test of Kwiatkowski et al. (1992). Consequently, a number of authors have considered algorithms for evaluating the distribution function of such random variables (see, e.g., the references in Forchini, 2002, Section 2.2). Typically, these algorithms involve inverting the characteristic function of the random variable of interest via the inversion formula of Gil-Pelaez (1951). Among them, Imhof's (1961) result appears to be the most widely used.

Unfortunately, the only case in which the characteristic function of a quadratic form is tractable is that in which the random vector entering it is Gaussian. In many applications, this assumption is too restrictive. The only paper of which we are aware that dispenses with this assumption is that of Glasserman et al. (2002). In it, the authors show how to evaluate the distribution function of a quadratic form in a multivariate t random vector. Their device is to express the probability of interest in terms of a certain auxiliary random variable, which, unlike the quadratic form of interest, possesses a tractable characteristic function. The first contribution of the present paper is the generalization of their result to the entire class of multivariate generalized hyperbolic distributions. As an application, we consider the distribution of the two stage least squares estimator of a structural parameter in a simultaneous system of equations.

The second contribution of the paper concerns partial moments. Consider a random variable X with finite first moment. The quantity $\mathbb{E}[X \mid X \leq x]$, that is, the expectation of X, conditional on falling into its own tail, has received considerable interest in the literature recently. In risk management, if X denotes the return on a financial position whose distribution is continuous at its qth quantile x_q, $-\mathbb{E}[X \mid X \leq x_q]$ is known as the expected shortfall, conditional value at risk, or tail conditional expectation, depending on author and context. It is slated to replace Value at Risk as the mandated measure of market risk to be used by banks for the purpose of determining regulatory capital requirements (Basel Committee on Banking Supervision, 2012, p. 3), not least because unlike Value at Risk, it defines a coherent risk measure in the sense of Artzner et al. (1999). The random variable of interest is often characterized most conveniently in terms of its characteristic function, possibly because its distribution arises from a convolution, as occurs in forming portfolios. Therefore, it is desirable to express the tail conditional expectation in terms of the characteristic function directly. As such, a number of authors have obtained expressions that facilitate such a computation; examples are Martin (2006), Kim et al. (2009), Broda & Paolella (2009), Pinelis (2010), Bormetti et al. (2010), and Feng & Lin (2013). The representation of the moment generating function of a truncated random variable given in Butler & Wood (2004) can also be used for this purpose. The aforementioned results all require, however, that the characteristic function be analytic in a strip containing the real axis, implying the existence of a moment generating function and hence all moments. They therefore fail for the generalized hyperbolic, except in special cases. The second contribution of the present paper is to provide an expression that is valid without such a restriction; in fact, in Section 2.1 below, a more general result will be proven which expresses the nth partial moment of a potentially heavy-tailed random variable X, provided it exists, in terms of the characteristic function. The result is a direct generalization of the inversion formula for the distribution function derived in Gil-Pelaez (1951), to which it collapses for $n = 0$.

In Section 2.2, the result is generalized to ratios of random variables. This is the third contribution of the paper, and is of interest because numerous common distributions, such as the Student's t, permit a stochastic representation of this form. In particular, it allows us to derive an expression for the partial expectation of a quadratic form in a multivariate generalized hyperbolic random vector in Section 3, thus generalizing the results of Yueh & Wong (2010) and Broda (2012), which deal with the Gaussian and multivariate t cases, respectively. As an application, we consider the expected shortfall of a quadratic portfolio, as arises from a delta-gamma approximation.

2 Inversion Formulae for Partial Moments

2.1 General Case

Let $F(x)$ denote the distribution function of X. For $n \in \mathbb{N}_0$, define $G_n(x) \equiv \int_{-\infty}^{x} x^n \mathrm{d}F(x)$, so that $F(x) \equiv G_0(x)$, and observe that at every point of continuity of F (and hence G_n),

$$\mathbb{E}[X^n \mid X \leq x] = \frac{G_n(x)}{F(x)}.$$

The following result will be proven.

Theorem 1 (Inversion Formula for Partial Moments)**.** *If the nth moment of X, $n \in \mathbb{N}_0$, is finite and $F(x)$ is continuous at x, then*

$$G_n(x) = \frac{\varphi^{(n)}(0)}{2\mathrm{i}^n} - \frac{1}{\pi}\int_0^\infty \mathrm{Im}\left[\frac{\mathrm{e}^{-\mathrm{i}tx}\varphi^{(n)}(t)}{\mathrm{i}^n t}\right]\mathrm{d}t,$$

where $\varphi^{(n)}(t)$ is the nth derivative of the characteristic function of X.

Proof. Denote by $\varphi(t)$ the characteristic function of X. If the nth moment of X is finite, then from Corollary 2 to Theorem 2.3.1 of Lukacs (1970),

$$\varphi^{(n)}(t) = \mathrm{i}^n \int_{-\infty}^{\infty} x^n e^{\mathrm{i}tx}\mathrm{d}F(x).$$

As in Gil-Pelaez (1951), define

$$\mathrm{sign}(y-x) = \frac{2}{\pi}\int_0^\infty \frac{\sin t(y-x)}{t}\mathrm{d}t = \begin{cases} -1, & y < x, \\ 0, & y = x, \\ 1, & y > x, \end{cases}$$

and observe that

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{sign}(y-x) y^n \mathrm{d}F(y) &= \int_x^\infty y^n \mathrm{d}F(y) - \int_{-\infty}^{x} y^n \mathrm{d}F(y) \\ &= \left[\mathbb{E}\left[X^n\right] - G_n(x)\right] - G_n(x) = \frac{\varphi^{(n)}(0)}{\mathrm{i}^n} - 2G_n(x).\end{aligned}$$

Then for $0 < \epsilon < T$,

$$\begin{aligned}\frac{1}{\pi}\int_\epsilon^T \frac{\mathrm{e}^{-\mathrm{i}tx}\varphi^{(n)}(t) - \mathrm{e}^{\mathrm{i}tx}\varphi^{(n)}(-t)}{\mathrm{i}^{n+1}t}\mathrm{d}t &= \frac{2}{\pi}\int_\epsilon^T \int_{-\infty}^{\infty} \frac{\sin(t(y-x))}{t} y^n \mathrm{d}F(y)\mathrm{d}t \\ &= \frac{2}{\pi}\int_{-\infty}^{\infty}\int_\epsilon^T \frac{\sin(t(y-x))}{t}\mathrm{d}t y^n \mathrm{d}F(y),\end{aligned}$$

where the exchange of the order of integration is permissible because for each fixed value of y,

$$\left|\frac{\sin t(y-x)}{t}\right| < \frac{1}{\epsilon}.$$

It remains to take the limit for $\epsilon \to 0$ and $T \to \infty$. Because the integral in t is a continuous function of ϵ and T with bounded modulus, one may pass the limit through the integral sign, to find

$$\begin{aligned}\lim_{\substack{\epsilon\to 0\\ T\to\infty}} \frac{1}{\pi}\int_{\epsilon}^{T} \frac{\mathrm{e}^{-\mathrm{i}tx}\varphi^{(n)}(t)-\mathrm{e}^{\mathrm{i}tx}\varphi^{(n)}(-t)}{\mathrm{i}^{n+1}t}\mathrm{d}t &= \frac{2}{\pi}\int_{-\infty}^{\infty}\lim_{\substack{\epsilon\to 0\\ T\to\infty}}\int_{\epsilon}^{T}\frac{\sin(t(y-x))}{t}\mathrm{d}t y^n \mathrm{d}F(y)\\ &= \int_{-\infty}^{\infty}\operatorname{sign}(y-x)y^n\mathrm{d}F(y) = \frac{\varphi^{(n)}(0)}{\mathrm{i}^n} - 2G_n(x).\end{aligned}$$

Thus

$$G_n(x) = \frac{\varphi^{(n)}(0)}{2\mathrm{i}^n} - \frac{1}{2\pi}\int_0^{\infty}\frac{\mathrm{e}^{-\mathrm{i}tx}\varphi^{(n)}(t)-\mathrm{e}^{\mathrm{i}tx}\varphi^{(n)}(-t)}{\mathrm{i}^{n+1}t}\mathrm{d}t.$$

The result follows upon noting that $\varphi^{(n)}(t)/\mathrm{i}^n$ is the complex conjugate of $\varphi^{(n)}(-t)/\mathrm{i}^n$. □

A few remarks are in order. First, the integral in the theorem does not, in general, converge absolutely, and must be interpreted as an improper Riemann integral. This is similar to the inversion integral of Gil-Pelaez (1951) for the distribution function, as remarked by Wendel (1961). It may be shown however that the integral converges absolutely under the Rosén-type condition (Rosén, 1961)

$$\int_{-\infty}^{\infty}\log(1+|x|)|x|^n\mathrm{d}F(x) < \infty.$$

Finally, Theorem 1 will clearly be most useful in situations where the integral permits no analytical solution, and must be evaluated by means of numerical quadrature schemes. In such cases, the doubly exponential transformation of Ooura & Mori (1991) may benefit the numerics.

2.2 Ratios of Random Variables

Consider a bivariate random variable (X_1, X_2) and let $R \equiv X_1/X_2$. The following theorem will be proven.

Theorem 2 (Partial Expectation of a Ratio)**.** *If (i) φ_{X_1,X_2} is integrable, (ii)* $\mathrm{pr}(X_2 > 0) = 1$, *(iii)* $\mathbb{E}\left[|X_1|^2\right] < \infty$*, and (iv)* $\mathbb{E}\left[X_2^{-2}\right] < \infty$*, then*

$$\mathbb{E}\left[R\mathbf{1}_{R<r}\right] = \frac{\varphi_{s0}(0,0)}{2} - \frac{1}{\pi}\int_0^{\infty}\operatorname{Im}\left[\varphi_{s0}(s,-rs)\right]\frac{\mathrm{d}s}{s},$$

where

$$\varphi_{s0}(s,t) \equiv \int_{-\infty}^{t}\frac{\partial}{\partial s}\varphi_{X_1,X_2}(s,t')\mathrm{d}t'.$$

Proof. Integrability of the characteristic function ensures that (X_1, X_2) has a density, which will be denoted as $f_X(x_1,x_2)$. By Hölder's inequality, (iii) and (iv) imply that $\mathbb{E}\left[|R|\right] < \infty$. Using that X_2 is almost surely positive,

$$\begin{aligned}\mathbb{E}\left[R\mathbf{1}_{R<r}\right] &= \mathbb{E}\left[\frac{X_1}{X_2}\mathbf{1}_{X_1<rX_2}\right] = \mathbb{E}\left[\left(X_1 - rX_2\right)X_2^{-1}\mathbf{1}_{X_1<rX_2} + r\mathbf{1}_{X_1<rX_2}\right]\\ &= \int_0^{\infty}\int_{-\infty}^{rx_2}\left(x_1 - rx_2\right)x_2^{-1}f_X(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2 + r\,\mathrm{pr}\left(X_1 < rX_2\right). \qquad (1)\end{aligned}$$

Consider a new random variable (Y_1, Y_2) with density

$$f_Y(y_1, y_2) = \frac{y_2^{-1}}{\mu_{-1}} f_X(y_1, y_2),$$

where $\mu_{-1} \equiv \mathbb{E}[X_2^{-1}]$ is finite by assumption. Then

$$\begin{aligned}\int_0^\infty \int_{-\infty}^{rx_2} (x_1 - rx_2)\, x_2^{-1} f_X(x_1, x_2) \mathrm{d}x_1 \mathrm{d}x_2 &= \mu_{-1} \int_0^\infty \int_{-\infty}^{ry_2} (y_1 - ry_2)\, f_Y(y_1, y_2) \mathrm{d}y_1 \mathrm{d}y_2 \\ &= \mu_{-1} \mathbb{E}\left[W_r \mathbf{1}_{W_r < 0}\right], \end{aligned} \tag{2}$$

where $W_r \equiv Y_1 - rY_2$. The characteristic function of W_r is

$$\varphi_{W_r}(s) = \varphi_{Y_1,Y_2}(s, -rs).$$

Here, φ_{Y_1,Y_2} denotes the joint characteristic function of (Y_1, Y_2), which is

$$\begin{aligned}\varphi_{Y_1,Y_2}(s,t) &= \int_0^\infty \int_{-\infty}^\infty e^{\mathrm{i}sy_1 + \mathrm{i}ty_2} f_Y(y_1, y_2) \mathrm{d}y_1 \mathrm{d}y_2 = \int_0^\infty \int_{-\infty}^\infty e^{\mathrm{i}sy_1 + \mathrm{i}ty_2} \frac{y_2^{-1}}{\mu_{-1}} f_X(y_1, y_2) \mathrm{d}y_1 \mathrm{d}y_2 \\ &= \frac{1}{\mu_{-1}} \int_0^\infty \mathrm{d}y_1 \int_{-\infty}^\infty \mathrm{d}y_2 f_X(y_1, y_2) \left[e^{\mathrm{i}sy_1 + \mathrm{i}ly_2} y_2^{-1} + \mathrm{i} \int_l^t e^{\mathrm{i}sy_1 + \mathrm{i}t' y_2} \mathrm{d}t' \right] \\ &= \varphi_{Y_1,Y_2}(s, l) + \frac{\mathrm{i}}{\mu_{-1}} \int_l^t \varphi_{X_1,X_2}(s, t') \mathrm{d}t',\end{aligned}$$

for any arbitrary but finite l. Taking the limit as $l \to -\infty$,

$$\varphi_{Y_1,Y_2}(s,t) = \frac{\mathrm{i}}{\mu_{-1}} \int_{-\infty}^t \varphi_{X_1,X_2}(s, t') \mathrm{d}t'$$

by the multivariate Riemann-Lebesgue lemma (see, e.g., Stein & Weiss, 1971, Theorem 1.2). Thus,

$$\varphi_{W_r}(s) = \frac{\mathrm{i}}{\mu_{-1}} \int_{-\infty}^{-rs} \varphi_{X_1,X_2}(s, t) \mathrm{d}t.$$

Finiteness of $\mathbb{E}[X_1]$ implies that $\varphi_{X_1,X_2}(s,t)$ is differentiable with respect to s, and an application of Leibniz' rule shows the derivative of $\varphi_{W_r}(s)$ to be

$$\begin{aligned}\varphi'_{W_r}(s) &\equiv \frac{\mathrm{i}}{\mu_{-1}} \frac{\mathrm{d}}{\mathrm{d}s} \int_{-\infty}^{-rs} \varphi_{X_1,X_2}(s,t) \mathrm{d}t = \frac{\mathrm{i}}{\mu_{-1}} \left[-r\varphi_{X_1,X_2}(s, -rs) + \int_{-\infty}^{-rs} \frac{\partial}{\partial s} \varphi_{X_1,X_2}(s,t) \mathrm{d}t \right] \\ &= \frac{\mathrm{i}}{\mu_{-1}} \left[-r\varphi_{X_1,X_2}(s, -rs) + \varphi_{s0}(s, -rs) \right].\end{aligned} \tag{3}$$

Combining (1), (2), (3) and using Theorem 1,

$$\begin{aligned}\mathbb{E}\left[R \mathbf{1}_{R<r}\right] &= \mu_{-1} \mathbb{E}\left[W_r \mathbf{1}_{W_r<0}\right] + r\,\mathrm{pr}\left(X_1 < rX_2\right) \\ &= \mu_{-1} \left[\frac{\varphi'_{W_r}(0)}{2\mathrm{i}} + \frac{1}{\pi} \int_0^\infty \mathrm{Re}\left[\varphi'_{W_r}(s)\right] \frac{\mathrm{d}s}{s} \right] + r\,\mathrm{pr}\left(X_1 < rX_2\right) \\ &= -\frac{r}{2} + \frac{\varphi_{s0}(0,0)}{2} + \frac{\mu_{-1}}{\pi} \int_0^\infty \mathrm{Re}\left[\varphi'_{W_r}(s)\right] \frac{\mathrm{d}s}{s} + r\,\mathrm{pr}\left(X_1 < rX_2\right).\end{aligned}$$

Next,

$$\frac{\mu_{-1}}{\pi}\int_0^\infty \mathrm{Re}\left[\varphi'_{W_r}(s)\right]\frac{\mathrm{d}s}{s} = -\frac{1}{\pi}\int_0^\infty \mathrm{Im}\left[-r\varphi_{X_1,X_2}(s,-rs)+\varphi_{s0}(s,-rs)\right]\frac{\mathrm{d}s}{s}$$
$$= \frac{r}{2} - r\,\mathrm{pr}\left(X_1 < rX_2\right) - \frac{1}{\pi}\int_0^\infty \mathrm{Im}\left[\varphi_{s0}(s,-rs)\right]\frac{\mathrm{d}s}{s},$$

where the last equality follows from Theorem 1 with $n = 0$. □

We immediately have the following corollary.

Corollary 3 (Mean of a Ratio)**.** *Under the conditions of Theorem 2,*

$$\mathbb{E}\left[R\right] = \int_0^\infty \left[\frac{\partial}{\partial s}\varphi_{X_1,X_2}(s,-t)\right]_{s=0}\mathrm{d}t$$

whenever the expectation exists.

Proof. An argument analogous to the one which led to (2) shows that

$$\mathbb{E}\left[R\right] = \mu_{-1}\mathbb{E}\left[W_r\right] + r.$$

Set $r = 0$ for simplicity. Then, using (3),

$$\mathbb{E}\left[R\right] = \mu_{-1}\mathbb{E}\left[W_0\right] = \mu_{-1}(-\mathrm{i})\varphi'_{W_0}(0) = \varphi_{s0}(0,0) = \int_{-\infty}^0 \left[\frac{\partial}{\partial s}\varphi_{X_1,X_2}(s,t)\right]_{s=0}\mathrm{d}t.$$

A change of variables gives the result. □

Corollary 3 partially generalizes Lemma 1 of Sawa (1972). Sawa's result applies to higher order moments, but requires the existence of the joint c.f. for purely imaginary arguments, which is not assumed here.

3 Quadratic Forms

3.1 Tail Probabilities

Consider the random variable

$$L \equiv a_0 + \mathbf{a}^{\mathrm{T}}X + X^{\mathrm{T}}\mathbf{A}X, \tag{4}$$

a quadratic plus a linear form in the random vector X. Suppose $X \sim \mathrm{MGHyp}(\boldsymbol{\mu}, \mathbf{C}, \boldsymbol{\gamma}, \lambda, \chi, \psi)$; that is, X has a d-variate generalized hyperbolic distribution with stochastic representation

$$X = \boldsymbol{\mu} + Y\boldsymbol{\gamma} + \sqrt{Y}\mathbf{C}Z, \tag{5}$$

where Z has a d-variate standard Normal distribution, $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ are constant d-vectors, $\mathbf{C}$ is a $d \times d$ matrix, and Y has a univariate generalized inverse Gaussian distribution with density

$$f_{GIG}(y;\lambda,\chi,\psi) \equiv \frac{y^{\lambda-1}}{k_\lambda(\chi,\psi)}\exp\left\{-\frac{1}{2}\left(\chi y^{-1}+\psi y\right)\right\},$$

where

$$k_\lambda(\chi,\psi) \equiv \begin{cases} \frac{\psi}{2}^{-\lambda}\Gamma(\lambda), \text{ if } \chi = 0 \\ \frac{\chi}{2}^{\lambda}\Gamma(-\lambda), \text{ if } \psi = 0 \\ 2\left(\frac{\chi}{\psi}\right)^{\lambda/2} K_\lambda(\sqrt{\chi\psi}), \text{ if } \chi \neq 0 \text{ and } \psi \neq 0. \end{cases}$$

Here, $K_\lambda(z)$ is the modified Bessel function of the second kind of order ν, which, for $\mathrm{Re}(z) > 0$, has the integral representation

$$K_\lambda(z) \equiv \frac{1}{2}\int_0^\infty t^{\lambda-1} \exp\left\{-\frac{1}{2}z\left(t+t^{-1}\right)\right\} \mathrm{d}z. \tag{6}$$

Unlike in the Gaussian case, the characteristic function of L is intractable, so that standard results concerning inversion of characteristic functions as used by Imhof (1961) fail. Instead, let $Q \equiv L - a_0 - \mathbf{a}^\mathrm{T}\boldsymbol{\mu} - \boldsymbol{\mu}^\mathrm{T}\mathbf{A}\boldsymbol{\mu}$ and consider the auxiliary random variable

$$Q_0 \equiv \frac{Q}{Y} = \mathbf{a}^\mathrm{T}\boldsymbol{\gamma} + Z^\mathrm{T}\mathbf{C}^\mathrm{T}\mathbf{A}\mathbf{C}Z + 2\boldsymbol{\mu}^\mathrm{T}\mathbf{A}\boldsymbol{\gamma} + \frac{1}{\sqrt{Y}}\left(\mathbf{a}^\mathrm{T}\mathbf{C}Z + 2\boldsymbol{\mu}^\mathrm{T}\mathbf{A}\mathbf{C}Z\right) + \sqrt{Y}\left(2\boldsymbol{\gamma}\mathbf{A}\mathbf{C}Z\right) + Y\boldsymbol{\gamma}^\mathrm{T}\mathbf{A}\boldsymbol{\gamma}.$$

It will turn out that unlike those of L and Q, the joint characteristic function of Q_0 and Y^{-1}, $\varphi_{Q_0,Y^{-1}}$ say, is tractable, so that the cdf can be evaluated using the classical inversion formula of Gil-Pelaez (1951), as follows:

$$\begin{aligned} \mathrm{pr}[Q \leq x] &= \mathrm{pr}[Q_0 Y \leq x] = \mathrm{pr}[Q_0 - xY^{-1} \leq 0] \\ &= \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Im}\left[\varphi_{Q_0,Y^{-1}}(s,-sx)\right]\frac{\mathrm{d}s}{s}. \end{aligned} \tag{7}$$

We begin by constructing the spectral decomposition

$$\mathbf{P}\boldsymbol{\Lambda}\mathbf{P}^\mathrm{T} = \mathbf{C}^\mathrm{T}\mathbf{A}\mathbf{C},$$

where $\boldsymbol{\Lambda}$ is diagonal with entries $\lambda_j, j \in \{1,\ldots,d\}$, the eigenvalues of $\mathbf{C}^\mathrm{T}\mathbf{A}\mathbf{C}$, and $\mathbf{P}$ is orthogonal. Then $Z^\mathrm{T}\mathbf{C}^\mathrm{T}\mathbf{A}\mathbf{C}Z =_d \sum_{j=1}^d \lambda_j Z_j^2$. Further define, for notational convenience, $c \equiv \mathbf{a}^\mathrm{T}\boldsymbol{\gamma} + 2\boldsymbol{\mu}^\mathrm{T}\mathbf{A}\boldsymbol{\gamma}$, $\mathbf{d} = \mathbf{a}^\mathrm{T}\mathbf{C}\mathbf{P} + 2\boldsymbol{\mu}^\mathrm{T}\mathbf{A}\mathbf{C}\mathbf{P}$, $\mathbf{e} = 2\boldsymbol{\gamma}\mathbf{A}\mathbf{C}\mathbf{P}$, and $k = \boldsymbol{\gamma}^\mathrm{T}\mathbf{A}\boldsymbol{\gamma}$. Denote by d_j and e_j the individual elements of $\mathbf{d}$ and $\mathbf{e}$, respectively. We then have the following result.

Theorem 4 (Distribution of Quadratic Form)**.** *Let* $L \equiv a_0 + \mathbf{a}^\mathrm{T}X + X^\mathrm{T}\mathbf{A}X$*, where* $X \sim \mathrm{MGHyp}(\boldsymbol{\mu}, \mathbf{C}, \boldsymbol{\gamma}, \lambda, \chi, \psi)$. *Then*

$$\mathrm{pr}[L \leq l] = \frac{1}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Im}\left[\Xi_\lambda(s, -xs, \chi, \psi)\right]\frac{\mathrm{d}s}{s}, \tag{8}$$

where $x = l - a_0 - \mathbf{a}^\mathrm{T}\boldsymbol{\mu} - \boldsymbol{\mu}^\mathrm{T}\mathbf{A}\boldsymbol{\mu}$,

$$\Xi_\lambda(s,t,\chi,\psi) \equiv \frac{k_\lambda(\chi - 2\alpha_2(s) - 2\mathrm{i}t, \psi - 2\alpha_1(s))}{k_\lambda(\chi,\psi)}\rho(s),$$

$$\alpha_1(s) \equiv \mathrm{i}ks - \frac{1}{2}s^2\sum_{j=1}^d \frac{e_j^2}{1-2\mathrm{i}s\lambda_j}, \quad \alpha_2(s) \equiv -\frac{1}{2}s^2\sum_{j=1}^d \frac{d_j^2}{1-2\mathrm{i}s\lambda_j},$$

$$\text{and } \rho(s) \equiv \exp\left\{\mathrm{i}sc - s^2\sum_{j=1}^d \frac{d_j e_j}{1-2\mathrm{i}s\lambda_j}\right\}\prod_{j=1}^d \frac{1}{\sqrt{1-2\mathrm{i}s\lambda_j}}.$$

Proof. Combining (4) and (5),

$$L = a_0 + \mathbf{a}^{\mathrm{T}}\boldsymbol{\mu} + \boldsymbol{\mu}^{\mathrm{T}}\mathbf{A}\boldsymbol{\mu} + \sqrt{Y}\left(\mathbf{a}^{\mathrm{T}}\mathbf{C}Z + 2\boldsymbol{\mu}^{\mathrm{T}}\mathbf{A}\mathbf{C}Z\right) + Y\left(\mathbf{a}^{\mathrm{T}}\boldsymbol{\gamma} + Z^{\mathrm{T}}\mathbf{C}^{\mathrm{T}}\mathbf{A}\mathbf{C}Z + 2\boldsymbol{\mu}^{\mathrm{T}}\mathbf{A}\boldsymbol{\gamma}\right) \\ + Y^{3/2}\left(2\boldsymbol{\gamma}\mathbf{A}\mathbf{C}Z\right) + Y^2\boldsymbol{\gamma}^{\mathrm{T}}\mathbf{A}\boldsymbol{\gamma}.$$

Write

$$Q_0 =_d c + kY + \sum_{j=1}^{d}\left(d_j\frac{1}{\sqrt{Y}} + e_j\sqrt{Y}\right)Z_j + \lambda_j Z_j^2.$$

The characteristic function of Q_0, conditional on Y, is

$$\begin{aligned}\mathbb{E}\left[e^{\mathrm{i}sQ_0} \mid Y\right] &= e^{\mathrm{i}s(c+kY)}\mathbb{E}\left[\exp\left\{\mathrm{i}s\sum_{j=1}^{d}\left(d_j\frac{1}{\sqrt{Y}} + e_j\sqrt{Y}\right)Z_j + \lambda_j Z_j^2\right\}\right] \\ &= e^{\alpha_1(s)Y + \alpha_2(s)Y^{-1}}\rho(s).\end{aligned}$$

We will require the joint characteristic function of Q_0 and Y^{-1},

$$\begin{aligned}\varphi_{Q_0,Y^{-1}}(s,t) \equiv \mathbb{E}\left[e^{\mathrm{i}sQ_0+\mathrm{i}tY^{-1}}\right] &= \mathbb{E}\left[\mathbb{E}\left[e^{\mathrm{i}sQ_0+\mathrm{i}tY^{-1}} \mid Y\right]\right] \\ &= \rho(s)\mathbb{E}\left[e^{\alpha_1(s)Y+[\alpha_2(s)+\mathrm{i}t]Y^{-1}}\right].\end{aligned}$$

Using the expression for the density of the generalized inverse Gaussian,

$$\begin{aligned}\mathbb{E}\left[e^{\alpha_1(s)Y+[\alpha_2(s)+\mathrm{i}t]Y^{-1}}\right] &= \int_0^\infty e^{\alpha_1(s)y+[\alpha_2(s)+\mathrm{i}t]y^{-1}} f_{GIG}(y;\lambda,\chi,\psi)\mathrm{d}y \\ &= \int_0^\infty e^{\alpha_1(s)y+[\alpha_2(s)+\mathrm{i}t]y^{-1}}\frac{y^{\lambda-1}}{k_\lambda(\chi,\psi)}\exp\left\{-\frac{1}{2}\left(\chi y^{-1}+\psi y\right)\right\}\mathrm{d}y \\ &= \frac{k_\lambda(\chi-2\alpha_2(s)-2\mathrm{i}t,\psi-2\alpha_1(s))}{k_\lambda(\chi,\psi)},\end{aligned}$$

so that

$$\varphi_{Q_0,Y^{-1}}(s,t) = \Xi_\lambda(s,t,\chi,\psi) \equiv \frac{k_\lambda(\chi-2\alpha_2(s)-2\mathrm{i}t,\psi-2\alpha_1(s))}{k_\lambda(\chi,\psi)}\rho(s). \tag{9}$$

Note that $\mathrm{Re}\big(\alpha_j(s)\big) < 0$, $j \in \{1,2\}$, so that together with the integral representation for $K_\lambda(z)$ given in (6), (9) is valid for all $(s,t) \in \mathbb{R}^2$.

Using (9) in (7) and undoing the location shift from Q to L gives the result. □

3.2 Partial Expectation

We now turn our attention to the partial expectation of L. The challenge in applying Theorem 2 is to find an analytic expression for $\varphi_{s0}(s,t) \equiv \int_{-\infty}^{t}\frac{\partial}{\partial s}\Xi_\lambda(s,t',\chi,\psi)\mathrm{d}t'$, where $\Xi_\lambda(s,t,\chi,\psi)$ is as in (9). Let

$$\Xi_\lambda^i(s,t,\chi,\psi) \equiv \frac{k_{\lambda+i}(\chi-2\alpha_2(s)-2\mathrm{i}t,\psi-2\alpha_1(s))}{k_\lambda(\chi,\psi)}\rho(s),$$

so that $\Xi_\lambda^0(s,t,\chi,\psi) \equiv \Xi_\lambda(s,t,\chi,\psi)$. We will need the following result.

Lemma 5. *The following relationships hold.*

1. $\mathrm{i}\int_{-\infty}^{t} \Xi_\lambda^i(s,t',\chi,\psi)\mathrm{d}t' = \Xi_\lambda^{i+1}(s,t,\chi,\psi)$.

2. $\frac{\partial}{\partial t}\Xi_\lambda^i(s,t,\chi,\psi) = \mathrm{i}\Xi_\lambda^{i-1}(s,t,\chi,\psi)$.

3. $\frac{\partial}{\partial s}\Xi_\lambda^i(s,t,\chi,\psi) = \frac{\mathrm{d}\log\rho(s)}{\mathrm{d}s}\Xi_\lambda^i(s,t,\chi,\psi) + \frac{\mathrm{d}\alpha_2(s)}{\mathrm{d}s}\Xi_\lambda^{i-1}(s,t,\chi,\psi) + \frac{\mathrm{d}\alpha_1(s)}{\mathrm{d}s}\Xi_\lambda^{i+1}(s,t,\chi,\psi)$.

Proof. Given in the Supplementary Material. □

With this result in hand, the following theorem follows at once.

Theorem 6 (Partial Expectation of Quadratic Form)**.** *Let* $L \equiv a_0 + \mathbf{a}^{\mathrm{T}}X + X^{\mathrm{T}}\mathbf{A}X$, $X \sim \mathrm{MGHyp}(\boldsymbol{\mu}, \mathbf{C}, \boldsymbol{\gamma}, \lambda, \chi, \psi)$. *Then*

$$\mathbb{E}[L\mathbf{1}_{L<l}] = (l-x)\,\mathrm{pr}[L<l] + \frac{\varphi_{s0}(0,0)}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Im}\left[\varphi_{s0}(s,-xs)\right]\frac{\mathrm{d}s}{s}, \tag{10}$$

where $x = l - a_0 - \mathbf{a}^{\mathrm{T}}\boldsymbol{\mu} - \boldsymbol{\mu}^{\mathrm{T}}\mathbf{A}\boldsymbol{\mu}$, $\varphi_{s0}(s,t) = \sum_{i=0}^{2}\Xi_\lambda^i(s,t,\chi,\psi)\beta_i(s)$,

$$\Xi_\lambda^i(s,t,\chi,\psi) \equiv \frac{k_{\lambda+i}(\chi - 2\alpha_2(s) - 2\mathrm{i}t, \psi - 2\alpha_1(s))}{k_\lambda(\chi,\psi)}\rho(s),$$

$\alpha_1(s)$, $\alpha_2(s)$, *and* $\rho(s)$ *are as in Theorem 4, and* $\beta_0(s)$, $\beta_1(s)$ *and* $\beta_2(s)$ *are defined in (11–13) below.*

Proof. Let $x = l - a_0 - \mathbf{a}^{\mathrm{T}}\boldsymbol{\mu} - \boldsymbol{\mu}^{\mathrm{T}}\mathbf{A}\boldsymbol{\mu}$ and $Q = L - a_0 - \mathbf{a}^{\mathrm{T}}\boldsymbol{\mu} - \boldsymbol{\mu}^{\mathrm{T}}\mathbf{A}\boldsymbol{\mu}$ as before. It is easy to see that

$$\mathbb{E}[L\mathbf{1}_{L<l}] = \mathbb{E}[Q\mathbf{1}_{Q<x}] + (l-x)\,\mathrm{pr}[L<l].$$

Using Theorem 2,

$$\mathbb{E}\left[Q\mathbf{1}_{Q<x}\right] = \frac{\varphi_{s0}(0,0)}{2} - \frac{1}{\pi}\int_0^\infty \mathrm{Im}\left[\varphi_{s0}(s,-xs)\right]\frac{\mathrm{d}s}{s},$$

where, using Lemma 5 twice,

$$\begin{aligned}\varphi_{s0}(s,t) &\equiv \int_{-\infty}^{t}\frac{\partial}{\partial s}\varphi_{Q_0,Y^{-1}}(s,t')\mathrm{d}t' \\ &= \frac{\partial}{\mathrm{i}\partial s}\Xi_\lambda^{i+1}(s,t,\chi,\psi) = \sum_{i=0}^{2}\Xi_\lambda^i(s,t,\chi,\psi)\beta_i(s),\end{aligned}$$

and

$$\beta_0(s) \equiv \frac{\mathrm{d}\alpha_2(s)}{\mathrm{i}\,\mathrm{d}s} = \mathrm{i}s\sum_{j=1}^{d}\frac{d_j^2}{1-2\mathrm{i}s\lambda_j} - s^2\sum_{j=1}^{d}\frac{\lambda_j d_j^2}{(1-2\mathrm{i}s\lambda_j)^2}, \tag{11}$$

$$\beta_1(s) \equiv \frac{\mathrm{d}\log\rho(s)}{\mathrm{i}\,\mathrm{d}s} = c + 2\mathrm{i}s\sum_{j=1}^{d}\frac{d_j e_j}{1-2\mathrm{i}s\lambda_j} - 2s^2\sum_{j=1}^{d}\frac{\lambda_j d_j e_j}{(1-2\mathrm{i}s\lambda_j)^2} + \sum_{j=1}^{d}\frac{\lambda_j}{(1-2\mathrm{i}s\lambda_j)^2}, \tag{12}$$

$$\text{and } \beta_2(s) \equiv \frac{\mathrm{d}\alpha_1(s)}{\mathrm{i}\,\mathrm{d}s} = k + \mathrm{i}s\sum_{j=1}^{d}\frac{e_j^2}{1-2\mathrm{i}s\lambda_j} - s^2\sum_{j=1}^{d}\frac{\lambda_j e_j^2}{(1-2\mathrm{i}s\lambda_j)^2}. \tag{13}$$

□

Implementations in Matlab and Fortran of Expressions (8) and (10) are available at the first author's Github page.[1] Evaluating them typically takes about a millisecond.

4 Applications

4.1 Sampling Distribution of the Two Stage Least Squares Estimator

The two-stage least squares estimator is used for obtaining causal inferences in simultaneous equation models, which often arise when randomized controlled trials are infeasible. This situation is common in econometrics, but also occurs in, e.g., molecular epidemiology (Thomas & Conti, 2004; Lawlor et al., 2008), in the context of Mendelian randomization. The sampling distribution of the two stage least squares estimator under normality has been the subject of intense study; early works include Richardson (1968), Sawa (1969), Anderson & Sawa (1973), and Holly & Phillips (1979). The realization that asymptotic normality of the estimator fails to hold when the instruments are only weakly correlated with the endogenous regressor has spurred renewed interest in the finite sample distribution, both in econometrics (Nelson & Startz, 1990a,b; Maddala & Jeong, 1992; Staiger & Stock, 1997; Woglom, 2001; Hillier, 2006; Forchini, 2006; Phillips, 2006) and epidemiology (Burgess et al., 2011; Burgess & Thompson, 2011). A related problem occurs when the number of instruments is large (Bekker, 1994; Davies et al., 2015). To the best of our knowledge, only three authors have considered the exact sampling distribution of the estimator under non-Gaussianity: Knight (1986) assumes that the error distribution is expandable in an Edgeworth-type expansion. Forchini (2007) assumes an elliptical law, but his results are limited to a just-identified equation, where one has a single instrument per endogenous regressor. The result of Broda & Kan (2015) applies to a large class of distributions, but it, too, is limited to just-identified equations. Here, we express the estimator as a linear plus a quadratic form in the innovation vector, as in Cribbett et al. (1989). The results of the paper then facilitate the computation of the exact sampling distribution under multivariate generalized hyperbolic innovations, for any number of weak or strong instruments, and with an arbitrary fixed covariance structure.

Consider the simultaneous equations model

$$\begin{aligned}\mathbf{y}_1 &= \mathbf{y}_2\beta + \mathbf{X}\boldsymbol{\gamma} + \mathbf{u}\\ \mathbf{y}_2 &= \mathbf{Z}_1\boldsymbol{\pi} + \mathbf{X}\boldsymbol{\delta} + \mathbf{v},\end{aligned}$$

where $\mathbf{y}_1 = (y_{1,1}, \ldots, y_{1,n})^{\mathrm{T}}$, $\mathbf{y}_2 = (y_{2,1}, \ldots, y_{2,n})^{\mathrm{T}}$, $\mathbf{u} = (u_1, \ldots, u_n)^{\mathrm{T}}$, $\mathbf{v} = (v_1, \ldots, v_n)^{\mathrm{T}}$, $\mathbf{X}$ is an $n \times k$ matrix of exogenous regressors, $\mathbf{Z}_1 \notin \mathcal{C}(\mathbf{X})$ is an $n \times k_1$ matrix of instruments assumed either fixed or conditioned upon, and

$$\begin{pmatrix} u_i \\ v_i \end{pmatrix} \sim \left(\mathbf{0}, \begin{bmatrix} \sigma_u^2 & \sigma_{uv} \\ \sigma_{uv} & \sigma_v^2 \end{bmatrix}\right),$$

so that

$$\boldsymbol{\varepsilon} \equiv \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix} \sim \left(\mathbf{0}, \begin{bmatrix} \sigma_u^2\mathbf{I} & \sigma_{uv}\mathbf{I} \\ \sigma_{uv}\mathbf{I} & \sigma_v^2\mathbf{I} \end{bmatrix}\right). \tag{14}$$

Let $\mathbf{M}_\mathbf{X} \equiv \mathbf{I} - \mathbf{X}(\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}$ and define $\mathbf{Z} \equiv \mathbf{M}_\mathbf{X}\mathbf{Z}_1$. The parameter of interest is β. If $\sigma_{uv} \neq 0$, then $\mathbf{X}$ is an endogenous regressor, and the ordinary least squares estimator of β is biased. A solution is to use the two stage least squares estimator for β, defined as

$$\hat{\beta}_{2SLS} = \frac{\mathbf{y}_2^{\mathrm{T}}\mathbf{P}_\mathbf{Z}\mathbf{y}_1}{\mathbf{y}_2^{\mathrm{T}}\mathbf{P}_\mathbf{Z}\mathbf{y}_2}, \tag{15}$$

[1] `https://github.com/s-broda/es4mgh`

where $\mathbf{P_Z} \equiv \mathbf{Z}(\mathbf{Z}^{\mathrm{T}}\mathbf{Z})^{-1}\mathbf{Z}^{\mathrm{T}}$. Let $\hat{B} = \hat{\beta}_{2SLS} - \beta$, the estimation error. Then

$$\hat{B} = \frac{\boldsymbol{\pi}^{\mathrm{T}}\mathbf{Z}^{\mathrm{T}}\mathbf{u} + \mathbf{v}^{\mathrm{T}}\mathbf{P_Z}\mathbf{u}}{\boldsymbol{\pi}^{\mathrm{T}}\mathbf{Z}^{\mathrm{T}}\mathbf{Z}\boldsymbol{\pi} + 2\boldsymbol{\pi}^{\mathrm{T}}\mathbf{Z}^{\mathrm{T}}\mathbf{v} + \mathbf{v}^{\mathrm{T}}\mathbf{P_Z}\mathbf{v}}. \tag{16}$$

As $\mathbf{P_Z}$ is positive semidefinite, the denominator in (15), and hence (16), is almost surely positive. Thus

$$\begin{aligned} \operatorname{pr}\left[\hat{B} \le b\right] &= \operatorname{pr}\left[\boldsymbol{\pi}^{\mathrm{T}}\mathbf{Z}^{\mathrm{T}}\mathbf{u} + \mathbf{v}^{\mathrm{T}}\mathbf{P_Z}\mathbf{u} - b\left(\boldsymbol{\pi}^{\mathrm{T}}\mathbf{Z}^{\mathrm{T}}\mathbf{Z}\boldsymbol{\pi} + 2\boldsymbol{\pi}^{\mathrm{T}}\mathbf{Z}^{\mathrm{T}}\mathbf{v} + \mathbf{v}^{\mathrm{T}}\mathbf{P_Z}\mathbf{v}\right) \le 0\right] \\ &= \operatorname{pr}\left[a_0 + \mathbf{a}^{\mathrm{T}}\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^{\mathrm{T}}\mathbf{A}\boldsymbol{\varepsilon} \le 0\right], \end{aligned} \tag{17}$$

where

$$a_0 \equiv -b\boldsymbol{\pi}^{\mathrm{T}}\mathbf{Z}^{\mathrm{T}}\mathbf{Z}\boldsymbol{\pi}, \quad \mathbf{a} \equiv \begin{pmatrix} \mathbf{Z}\boldsymbol{\pi} \\ -2b\mathbf{Z}\boldsymbol{\pi} \end{pmatrix}, \quad \text{and } \mathbf{A} \equiv \frac{1}{2}\begin{pmatrix} \mathbf{0} & \mathbf{P_Z} \\ \mathbf{P_Z} & -2b\mathbf{P_Z} \end{pmatrix}.$$

Equation (17) is in the required form, so that Theorem 4 is readily applied.

4.2 Risk Measures and Portfolio Optimization

Given a probability space $(\Omega, \mathcal{F}, \operatorname{pr})$, the expected shortfall at level $\alpha \in (0,1)$ of a financial position with terminal value $X : \Omega \to \mathbb{R}$ is defined as (Acerbi & Tasche, 2002a,b)

$$ES^{(\alpha)} \equiv -\frac{1}{\alpha}\left[\mathbb{E}[X\mathbf{1}_{X\le x^{(\alpha)}}] - x^{(\alpha)}(\operatorname{pr}[X \le x^{(\alpha)}] - \alpha)\right], \tag{18}$$

where $x^{(\alpha)} \equiv \inf\{x : \operatorname{pr}[X \le x] \ge \alpha\}$ is the $\alpha \times 100\%$ quantile of the distribution of X. Often $\alpha = 1\%$. Apart from the sign, $x^{(\alpha)}$ corresponds to the Value at Risk at level α, which is a risk measure in its own right. The definition in (18) ensures that $ES^{(\alpha)}$ is a coherent risk measure in the sense of Artzner et al. (1999) even if X is not absolutely continuous. In our setting, X is indeed absolutely continuous, so that the expected shortfall, expressed in terms of the loss $L = -X$, reduces to

$$ES_L^{(\alpha)} \equiv \frac{1}{\alpha}\left[\mathbb{E}[L\mathbf{1}_{L>l^{(1-\alpha)}}]\right], \tag{19}$$

where $\operatorname{pr}\left[L > l^{(1-\alpha)}\right] = \alpha$.

As in Broda (2012), consider a portfolio with associated loss $L = -X$, composed of stocks $S = (S_1, \ldots, S_d)^{\mathrm{T}}$ and options written on those stocks. The task is to find an optimal portfolio, expressed as a vector of portfolio weights w, which minimizes the expected shortfall under a constraint on the desired expected return, or equivalently, on the tolerable expected loss. Because the price of an option depends on the price of the underlying stock in a complicated nonlinear fashion, it is popular among risk managers to approximate the loss by a second-order Taylor expansion in $\Delta S \equiv S - S_0$, known as a Delta-Gamma-Theta approximation. For a given time horizon T, the portfolio loss then becomes

$$L = a_0 + \mathbf{a}^{\mathrm{T}}\Delta S + \Delta S^{\mathrm{T}}\mathbf{A}\Delta S, \tag{20}$$

where $a_0 = -T\Theta$, $\mathbf{a} = -\boldsymbol{\delta}$, and $\mathbf{A} = -\frac{1}{2}\boldsymbol{\Gamma}$. Here $\delta_j = \partial X/\partial S_j$, $\Gamma_{jk} = \partial^2 X/\partial S_j \partial S_k$, and $\Theta = -\partial X/\partial T$ are the so-called "Greeks". These are functions of the portfolio weights w, but we leave this dependence implicit.

As in Broda et al. (2017), define the check function

$$\Theta_\alpha(c, \mathbf{w}) \equiv c + \frac{1}{\alpha}\int_c^\infty (L - c) f_L(\ell; \mathbf{w})\mathrm{d}\ell. \tag{21}$$

Differentiating (21) with respect to c using Leibniz' rule shows that for given $\mathbf{w}$, $\Theta_\alpha(c, \mathbf{w})$ attains its unique minimum at $c = \ell^{(1-\alpha)}$, and that $\Theta_\alpha(\ell^{(1-\alpha)}) = \mathrm{ES}_L^{(\alpha)}$. Furthermore, Rockafellar & Uryasev (2000) show that the mean-expected shortfall portfolio optimization problem can be solved by minimizing (21) jointly with respect to c and $\mathbf{w}$. Hence, for a given tolerable expected loss τ, the mean-expected shortfall optimization problem is $\min_{c,\mathbf{w}} \Theta_\alpha(c,\mathbf{w})$ s.t. $\mathbb{E}[L(w)] \le \tau, \mathbf{w}^{\mathrm{T}}\mathbf{1} = 1$. Alexander et al. (2006) show that the quadratic mean-expected shortfall problem is, in general, not well defined. As a solution, they propose regularizing the problem by adding trading and management costs. The problem then becomes

$$\min_{c,\mathbf{w}} \Theta_\alpha(c,\mathbf{w}) + |\mathbf{w}|^{\mathrm{T}}\boldsymbol{\kappa} \quad \text{s.t. } \mathbb{E}[L(w)] \le \tau, \mathbf{w}^{\mathrm{T}}\mathbf{1} = 1, \quad \text{where} \tag{22}$$
$$\Theta_\alpha(c,\mathbf{w}) = c + \frac{1}{\alpha}\left(\mathbb{E}[L(w)\mathbf{1}_{L(w)>c}] - c\{1 - \mathrm{pr}[L(w) \le c]\}\right),$$

$|\cdot|$ is elementwise absolute value, and κ is a cost vector. Note that if the cost is identical for all assets and there are no short sales, then the cost term has no influence on the optimal portfolio weights, as in that case $|w|^{\mathrm{T}}\mathbf{1} = 1$.

As a numerical example, we consider a set of stocks from the Dow Jones Industrial Average. This index comprises 30 'blue chip' stocks, of which we select 28 for having prices available from January 01, 1987 to December 31, 2017. The parameters of the generalized hyperbolic distribution are calibrated via the EM algorithm, except that we fix $\lambda = -50$ because of the tendency of the likelihood to be flat in this parameter. The option sensitivities (Θ, δ, Γ) are obtained as derivatives of the pricing formula of Black & Scholes (1973), as is common in practice. We divide the stock price series into N sub-periods of equal length, and re-balance the portfolio at the beginning of each period, by solving (22) and evaluating $\mathbb{E}[L(w)\mathbf{1}_{L(w)>c}]$ and $\mathrm{pr}[L(w) \le c]$ using Theorems 4 and 6. For comparison, we apply a Monte Carlo method in which we dispense with the analytic expressions and evaluate $\mathbb{E}[L(w)\mathbf{1}_{L(w)>c}]$ and $\mathrm{pr}[L(w) \le c]$ in (22) by drawing 100,000 independent realizations from the calibrated MGHyp distribution.

We apply both methods across a variety of different combinations of option type (put/call), position (long/short), time to maturity ($T = 60, T = 180$), and whether delta hedging is applied. A subset of these experiments is shown in Table 2, with the remainder relegated to the Supplementary Material. In Table 2, we fix $\tau = \infty$ so that the optimizer of (22) corresponds to the global minimum expected shortfall portfolio. We also set $\kappa = 0$ because only long positions are considered. The results show that both methods result in optimized portfolios with very similar risk as measured by the Value and Risk and expected shortfall. The main difference is that the analytic method is more than 30 times faster.

Supplementary material

Supplementary material available online includes the proof of Lemma 5, a graphical illustration of the application in Section 4.1, and the results of additional experiments for the application in Section 4.2.

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Table 1: Performance of the analytic (A) and Monte Carlo (MC) methods. N, number of rebalancing periods; Opt, option type (put/call); Exp, days to expiry; D-H, delta-hedged; VaR/ES, Value at Risk/expected shortfall of optimized portfolio. Entries are averaged over the N re-balancing periods, with standard errors in parentheses. Calculations were performed in MATLAB® on an Intel® Xeon® X5570 workstation.

N	Pos	Opt	Exp	D-H	Algorithm Time		VaR and ES Values (%)			
					A	MC	A		MC	
							VaR	ES	VaR	ES
10	L	C	60	1	60.7s (27.5)	2185.0s (236.2)	5.75 (1.1)	6.91 (1.4)	6.31 (1.2)	7.58 (1.2)
10	L	C	60	0	40.7s (8.5)	2122.4s (240.9)	8.93 (3.0)	10.85 (3.7)	9.61 (3.3)	11.70 (3.3)
10	L	C	180	1	67.3s (34.3)	2205.0s (256.3)	5.75 (1.1)	6.91 (1.4)	6.25 (1.2)	7.52 (1.2)
10	L	C	180	0	42.2s (10.1)	2081.1s (214.3)	8.83 (2.9)	10.76 (3.6)	9.66 (3.3)	11.79 (3.3)
10	L	P	60	1	67.3s (34.0)	2212.2s (252.3)	5.74 (1.1)	6.90 (1.4)	6.30 (1.2)	7.57 (1.2)
10	L	P	60	0	48.5s (24.3)	1955.8s (229.9)	8.71 (2.7)	10.57 (3.3)	9.29 (2.9)	11.31 (2.9)
10	L	P	180	1	61.4s (27.3)	2158.2s (223.5)	5.73 (1.1)	6.89 (1.4)	6.24 (1.3)	7.50 (1.3)
10	L	P	180	0	40.2s (8.7)	2096.5s (219.3)	8.37 (2.4)	10.18 (3.0)	8.98 (2.6)	10.95 (2.6)

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A Proof of Lemma 5

The result is trivial if $\chi = 0$ or $\psi = 0$, so we only consider the case in which both χ and ψ are nonzero.

of Assertion 1. We have that

$$\begin{aligned} \mathrm{i} \int_{-\infty}^{t} \Xi_{\lambda}^{i}(s, t', \chi, \psi) \mathrm{d}t' &= \mathrm{i} \int_{-\infty}^{t} \frac{k_{\lambda+i}(\chi - 2\alpha_2(s) - 2\mathrm{i}t', \psi - 2\alpha_1(s))}{k_{\lambda}(\chi, \psi)} \rho(s) \mathrm{d}t' \\ &= \frac{2\mathrm{i}\rho(s)}{k_{\lambda}(\chi, \psi)} \int_{-\infty}^{t} \left(\frac{u(t')}{\psi - 2\alpha_1(s)} \right)^{\lambda+i} K_{\lambda+i}(u(t')) \mathrm{d}t', \end{aligned}$$

where $u(t) \equiv \sqrt{(\chi - 2\alpha_2(s) - 2\mathrm{i}t)(\psi - 2\alpha_1(s))}$, so that $\mathrm{d}t' = \mathrm{i}u(\psi - 2\alpha_1(s))^{-1}\mathrm{d}u$. Changing variables,

$$\begin{aligned} \frac{2\mathrm{i}\rho(s)}{k_{\lambda}(\chi, \psi)} \int_{-\infty}^{t} \left(\frac{u(t')}{\psi - 2\alpha_1(s)} \right)^{\lambda+i} K_{\lambda+i}(u(t')) \mathrm{d}t' &= \frac{-2\rho(s)}{k_{\lambda}(\chi, \psi)(\psi - 2\alpha_1(s))^{\lambda+i+1}} \int_{-\infty}^{u(t)} u^{\lambda+i+1} K_{\lambda+i}(u) \mathrm{d}u \\ &= \frac{2\rho(s)}{k_{\lambda}(\chi, \psi)} \left(\frac{u(t)}{\psi - 2\alpha_1(s)} \right)^{\lambda+i+1} K_{\lambda+i+1}(u(t)) \Leftrightarrow \end{aligned}$$

$$\mathrm{i} \int_{-\infty}^{t} \Xi_{\lambda}^{i}(s, t', \chi, \psi) \mathrm{d}t' = \Xi_{\lambda}^{i+1}(s, t, \chi, \psi), \tag{23}$$

where the penultimate equality follows from Abramowitz & Stegun (1964, Eq. 11.3.18). □

of Assertion 2. Differentiating (23) immediately yields

$$\frac{\partial}{\partial t} \Xi_{\lambda}^{i}(s, t, \chi, \psi) = \mathrm{i}\Xi_{\lambda}^{i-1}(s, t, \chi, \psi). \tag{24}$$

□

of Assertion 3. First observe that

$$\begin{aligned} \frac{\partial}{\partial \psi} k_{\lambda}(\chi, \psi) &= 2 \frac{\partial}{\partial \psi} \left(\frac{\chi}{\psi} \right)^{\lambda/2} K_{\lambda}(\sqrt{\chi\psi}) \\ &= 2 \left[\frac{1}{2\psi} \left(\frac{\chi}{\psi} \right)^{\lambda/2} K_{\lambda}'(\sqrt{\chi\psi}) \sqrt{\chi\psi} - \frac{\lambda}{2\psi} \left(\frac{\chi}{\psi} \right)^{\lambda/2} K_{\lambda}(\sqrt{\chi\psi}) \right] \\ &= \left(\frac{\chi}{\psi} \right)^{\lambda/2} \frac{1}{\psi} \left[-\sqrt{\chi\psi} K_{\lambda+1}(\sqrt{\chi\psi}) \right] = -\frac{1}{2} k_{\lambda+1}(\chi, \psi), \end{aligned} \tag{25}$$

where the penultimate equality follows from Abramowitz & Stegun (1964, Eq. 9.6.28). Then,

$$\begin{aligned}
\frac{\partial}{\partial s}\Xi_{\lambda}^{i}(s,t,\chi,\psi) &= \frac{\partial}{\partial s}\frac{k_{\lambda+i}(\chi-2\alpha_2(s)-2\mathrm{i}t,\psi-2\alpha_1(s))}{k_{\lambda}(\chi,\psi)}e^{\log(\rho(s))} \\
&= \frac{\mathrm{d}\log\rho(s)}{\mathrm{d}s}\Xi_{\lambda}^{i}(s,t,\chi,\psi)+\frac{\mathrm{d}\alpha_2(s)}{\mathrm{d}s}\frac{\partial}{\mathrm{i}\partial t}\Xi_{\lambda}^{i}(s,t,\chi,\psi) \\
&\qquad -\frac{2\rho(s)}{k_{\lambda}(\chi,\psi)}\frac{\mathrm{d}\alpha_1(s)}{\mathrm{d}s}\frac{\partial}{\partial\psi}k_{\lambda+i}(\chi-2\alpha_2(s)-2\mathrm{i}t,\psi-2\alpha_1(s)) \\
&= \frac{\mathrm{d}\log\rho(s)}{\mathrm{d}s}\Xi_{\lambda}^{i}(s,t,\chi,\psi)+\frac{\mathrm{d}\alpha_2(s)}{\mathrm{d}s}\Xi_{\lambda}^{i-1}(s,t,\chi,\psi) \\
&\qquad +\frac{\rho(s)}{k_{\lambda}(\chi,\psi)}\frac{\mathrm{d}\alpha_1(s)}{\mathrm{d}s}k_{\lambda+i+1}(\chi-2\alpha_2(s)-2\mathrm{i}t,\psi-2\alpha_1(s)) \\
&= \frac{\mathrm{d}\log\rho(s)}{\mathrm{d}s}\Xi_{\lambda}^{i}(s,t,\chi,\psi)+\frac{\mathrm{d}\alpha_2(s)}{\mathrm{d}s}\Xi_{\lambda}^{i-1}(s,t,\chi,\psi)+\frac{\mathrm{d}\alpha_1(s)}{\mathrm{d}s}\Xi_{\lambda}^{i+1}(s,t,\chi,\psi),
\end{aligned}$$

where the final equality follows by using (24) and (25). □

B Additional Material for Section 4.1

Figure 1 shows the substantial dependence of the distribution of the two stage least squares estimator on the degrees of freedom, ν, when the errors in equation (14) are drawn from a special case of the multivariate generalized hyperbolic: the Student's t distribution, scaled to have unit variance. We assume a just-identified model ($k_1 = 1$). In this case, the distribution of $\hat{\beta}_{2SLS}$ depends on Z and π only through the concentration parameter, $\mu \equiv \boldsymbol{\pi}^{\mathrm{T}}\mathbf{Z}^{\mathrm{T}}\mathbf{Z}\boldsymbol{\pi}/\sigma_v^2$, which we set to 0.5, corresponding to rather weak instruments. We further set $n = 25$, $\beta = 0$ and $\rho = 1$.

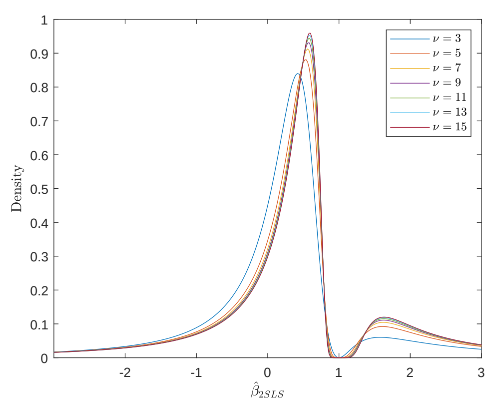

Figure 1: Density of two stage least squares estimator, obtained by numerical differentiation of the distribution function computed by applying Theorem 4 to (17).

C Additional Results for Section 4.2

Table 2: Performance of the analytic (A) and Monte Carlo (MC) methods. N, number of rebalancing periods; Opt, option type (put/call); Exp, days to expiry; D-H, delta-hedged; VaR/ES, Value at Risk/expected shortfall of optimized portfolio. Trading costs κ equal 0.1 for portfolios with short positions, 0 otherwise. Entries are averaged over the N re-balancing periods, with standard errors in parentheses. Calculations were performed in MATLAB® on an Intel® Xeon® X5570 workstation.

N	Pos	Opt	Exp	D-H	Algorithm Time		VaR and ES Values (%)			
					A	MC	A		MC	
							VaR	ES	VaR	ES
10	L	C	60	1	60.7s (27.5)	2185.0s (236.2)	5.75 (1.1)	6.91 (1.4)	6.31 (1.2)	7.58 (1.2)
10	L	C	60	0	40.7s (8.5)	2122.4s (240.9)	8.93 (3.0)	10.85 (3.7)	9.61 (3.3)	11.70 (3.3)
10	L	C	180	1	67.3s (34.3)	2205.0s (256.3)	5.75 (1.1)	6.91 (1.4)	6.25 (1.2)	7.52 (1.2)
10	L	C	180	0	42.2s (10.1)	2081.1s (214.3)	8.83 (2.9)	10.76 (3.6)	9.66 (3.3)	11.79 (3.3)
10	L	P	60	1	67.3s (34.0)	2212.2s (252.3)	5.74 (1.1)	6.90 (1.4)	6.30 (1.2)	7.57 (1.2)
10	L	P	60	0	48.5s (24.3)	1955.8s (229.9)	8.71 (2.7)	10.57 (3.3)	9.29 (2.9)	11.31 (2.9)
10	L	P	180	1	61.4s (27.3)	2158.2s (223.5)	5.73 (1.1)	6.89 (1.4)	6.24 (1.3)	7.50 (1.3)
10	L	P	180	0	40.2s (8.7)	2096.5s (219.3)	8.37 (2.4)	10.18 (3.0)	8.98 (2.6)	10.95 (2.6)
10	L	C&P	60	1	60.6s (26.8)	2203.7s (251.9)	5.75 (1.1)	6.91 (1.4)	6.32 (1.2)	7.59 (1.2)
10	L	C&P	60	0	36.4s (5.3)	2043.9s (219.9)	7.14 (1.4)	8.61 (1.7)	7.92 (1.8)	9.58 (1.8)
10	L	C&P	180	1	64.5s (30.3)	2220.5s (265.9)	5.74 (1.1)	6.89 (1.4)	6.24 (1.3)	7.51 (1.3)
10	L	C&P	180	0	41.7s (10.6)	1938.8s (237.7)	7.02 (1.3)	8.47 (1.6)	7.49 (1.4)	9.07 (1.4)
10	L&S	C	60	1	56.9s (23.8)	2193.4s (235.2)	5.75 (1.1)	6.91 (1.4)	6.34 (1.2)	7.61 (1.2)
10	L&S	C	60	0	40.8s (8.7)	2077.0s (218.7)	8.93 (3.0)	10.85 (3.7)	9.87 (3.5)	12.02 (3.5)
10	L&S	C	180	1	63.9s (31.9)	2202.1s (250.6)	5.75 (1.1)	6.91 (1.4)	6.27 (1.2)	7.54 (1.2)
10	L&S	C	180	0	38.0s (5.5)	2082.4s (215.6)	8.84 (2.9)	10.77 (3.6)	10.24 (4.1)	12.52 (4.1)
10	L&S	P	60	1	63.8s (29.0)	2172.5s (232.5)	5.74 (1.1)	6.90 (1.4)	6.31 (1.2)	7.58 (1.2)
10	L&S	P	60	0	37.5s (5.2)	2056.7s (214.0)	8.71 (2.7)	10.57 (3.3)	9.52 (3.1)	11.59 (3.1)
10	L&S	P	180	1	60.5s (26.2)	2119.8s (214.7)	5.74 (1.1)	6.89 (1.4)	6.24 (1.2)	7.51 (1.2)
10	L&S	P	180	0	38.7s (6.7)	2050.8s (227.3)	8.37 (2.4)	10.18 (3.0)	9.27 (2.9)	11.31 (2.9)
10	L&S	C&P	60	1	59.8s (26.9)	2113.3s (210.3)	5.74 (1.1)	6.90 (1.4)	6.31 (1.2)	7.57 (1.2)
10	L&S	C&P	60	0	37.4s (5.5)	2040.5s (207.1)	7.14 (1.4)	8.61 (1.7)	7.91 (1.8)	9.57 (1.8)
10	L&S	C&P	180	1	65.0s (30.7)	2191.6s (248.1)	5.74 (1.1)	6.89 (1.4)	6.26 (1.2)	7.52 (1.2)
10	L&S	C&P	180	0	37.7s (5.5)	2078.9s (222.1)	7.02 (1.3)	8.48 (1.6)	7.56 (1.5)	9.17 (1.5)
100	L	C	60	1	278.8s (12.9)	2555.3s (71.2)	0.86 (0.1)	0.86 (0.1)	2.06 (0.1)	2.07 (0.1)
100	L	C	60	0	53.6s (5.6)	1823.4s (70.1)	16.89 (1.0)	20.65 (1.3)	17.48 (1.1)	21.39 (1.1)
100	L	C	180	1	151.3s (8.0)	1486.5s (43.6)	0.85 (0.0)	0.85 (0.0)	1.41 (0.0)	1.42 (0.0)
100	L	C	180	0	39.6s (3.4)	1841.5s (75.5)	17.14 (1.1)	21.19 (1.4)	18.16 (1.2)	22.46 (1.2)
100	L	P	60	1	181.6s (9.3)	2070.1s (56.9)	0.82 (0.1)	0.82 (0.1)	1.86 (0.1)	1.87 (0.1)
100	L	P	60	0	37.9s (3.2)	1708.7s (62.3)	15.44 (1.0)	18.88 (1.2)	15.57 (1.0)	19.04 (1.0)
100	L	P	180	1	200.6s (12.6)	2020.1s (72.7)	0.69 (0.0)	0.69 (0.0)	1.19 (0.0)	1.20 (0.0)
100	L	P	180	0	43.3s (2.7)	1723.6s (74.8)	14.21 (0.9)	17.54 (1.2)	14.16 (0.9)	17.47 (0.9)
100	L	C&P	60	1	224.3s (12.0)	2703.2s (87.2)	0.85 (0.1)	0.85 (0.1)	1.90 (0.1)	1.92 (0.1)
100	L	C&P	60	0	37.7s (1.3)	1872.4s (75.7)	6.68 (0.3)	7.88 (0.4)	6.75 (0.4)	7.97 (0.4)
100	L	C&P	180	1	220.3s (12.5)	2699.3s (85.6)	0.79 (0.0)	0.80 (0.0)	1.30 (0.0)	1.31 (0.0)
100	L	C&P	180	0	30.2s (1.2)	1537.9s (55.6)	5.91 (0.3)	7.10 (0.4)	6.17 (0.3)	7.43 (0.3)
100	L&S	C	60	1	221.3s (11.2)	2687.9s (85.0)	0.86 (0.1)	0.86 (0.1)	2.11 (0.1)	2.12 (0.1)
100	L&S	C	60	0	48.8s (4.9)	1648.3s (61.8)	16.89 (1.0)	20.65 (1.3)	17.25 (1.1)	21.11 (1.1)
100	L&S	C	180	1	133.5s (7.6)	1472.1s (47.9)	0.85 (0.0)	0.85 (0.0)	1.42 (0.0)	1.43 (0.0)
100	L&S	C	180	0	48.7s (4.3)	1838.9s (67.3)	17.14 (1.1)	21.19 (1.4)	17.83 (1.2)	22.04 (1.2)
100	L&S	P	60	1	266.5s (12.6)	2559.1s (68.0)	0.82 (0.1)	0.82 (0.1)	1.83 (0.1)	1.84 (0.1)
100	L&S	P	60	0	41.4s (3.6)	1604.0s (64.1)	15.44 (1.0)	18.88 (1.2)	15.83 (1.0)	19.34 (1.0)
100	L&S	P	180	1	202.7s (12.4)	1986.0s (67.5)	0.69 (0.0)	0.69 (0.0)	1.21 (0.0)	1.21 (0.0)
100	L&S	P	180	0	24.1s (1.6)	1064.5s (53.5)	13.73 (0.9)	16.94 (1.1)	14.15 (0.9)	17.46 (0.9)
100	L&S	C&P	60	1	222.0s (11.8)	2685.3s (83.9)	0.85 (0.1)	0.85 (0.1)	1.93 (0.1)	1.95 (0.1)
100	L&S	C&P	60	0	31.2s (1.1)	1704.3s (59.1)	6.69 (0.3)	7.88 (0.4)	6.96 (0.4)	8.23 (0.4)
100	L&S	C&P	180	1	273.9s (15.3)	2565.5s (63.1)	0.79 (0.0)	0.80 (0.0)	1.32 (0.0)	1.33 (0.0)
100	L&S	C&P	180	0	35.4s (1.4)	2073.3s (65.9)	5.91 (0.3)	7.10 (0.4)	6.32 (0.4)	7.62 (0.4)